WE GET 2017

Students
Yohan Gourbière, Maelys Peltier, Sara Crespo, Claire Kewan, Sarah-Lou Morgen (lycée Arago) ; Andreea Vasile, Irina Pana, Veronica Sturzu (National College B. P. Hasdeu)

Schools
Lycée Arago (Perpignan, France), National College B. P. Hasdeu (Buzau, Romania)

Professors
Bogdan Enescu, Marie Diumenge, Christelle Quéru

Resarchers
Robert Brouzet, Bogdan Enescu

I) OUR PROJECT AND THE CHOICE OF THE SUBJECT

This year we had the opportunity to participate in MATh.en.JEANS, an exchange of experience between Romanian and French people*, where we have acquired a lot of information while working together as a team and discovering a universe where Mathematics offer us the chance to move between digits, but most importantly to see beyond the numbers. We have chosen a problem that inspired us and made us realise how many stories and relations could be hidden in some numbers, just if we let our minds play around with them and create new ones. Beyond each digit, each number, each equation there is an idea, waiting to be discovered. As we discover logically, we are closer to being able to move the Mathematics' world and finally see it differently.

* This work was done within the framework of the project Erasmus+ "Maths&Languages".

Our project consisted in choosing a problem from a list of subjects, being proposed to be solved during a few months. After a lot of attempts, we have finally reached the answer and we exposed our problem at the Mathematics conference, placed by ERASMUS in Montpellier to a grand audience whose attention we tried, as much as possible, to capture with our interesting method of solving it.

Our subject sounded like this: "*At the beginning you only have an empty blackboard. At each step you can either write twice the number 1 on the blackboard or delete two numbers equal to n already written and replace them by* $n-1$ *and* $n+1$. *How many steps at least will be necessary in order to reach the number* 2017*?*"

Not only we saw in this problem the opportunity to master mathematical knowledge, but we also saw the beauty of some simple numbers that can hide unexpectedly complex ideas. The most beautiful discoveries in Mathematics were hidden in just some simple ideas which have been found step by step, by moving each number and giving it a new meaning.

II) OUR SOLUTION

A. The "bad" idea

First of all, we thought that in order to reach 2017 we need 2 of 2016, because to reach n, we need 2 numbers of $n-1$. That is one step. In order to reach 2 of 2016, we need for each one 2 of 2015, meaning that we need 4 of 2015, and this means two steps. We continued with this idea until we wanted to reach 2 from 1. But how many numbers of 2 and how many numbers of 1 will be needed and how many steps are there?

We observed the next rule: to reach 2017 from 2016 there are 2^0 steps. To reach 2016 from 2015, there are 2^1 steps. Therefore, reaching n from $n-1$ takes $2^{2017-\mathrm{n}}$ steps.

To reach 2 from couples of 1, there are 2^{2015} steps and to write on the blackboard all the couples of 1, there are also 2^{2015} steps.

In order to find the total number of steps, we have to sum up all of the steps that we have done in order to get every number.

$$\text{steps: } 2^0 + 2^1 + 2^2 + \cdots + 2^{2015} + 2^{2015}$$

$$S = 2^0 + 2^1 + 2^2 + \cdots + 2^{2015} + 2^{2015} \mid \times 2$$

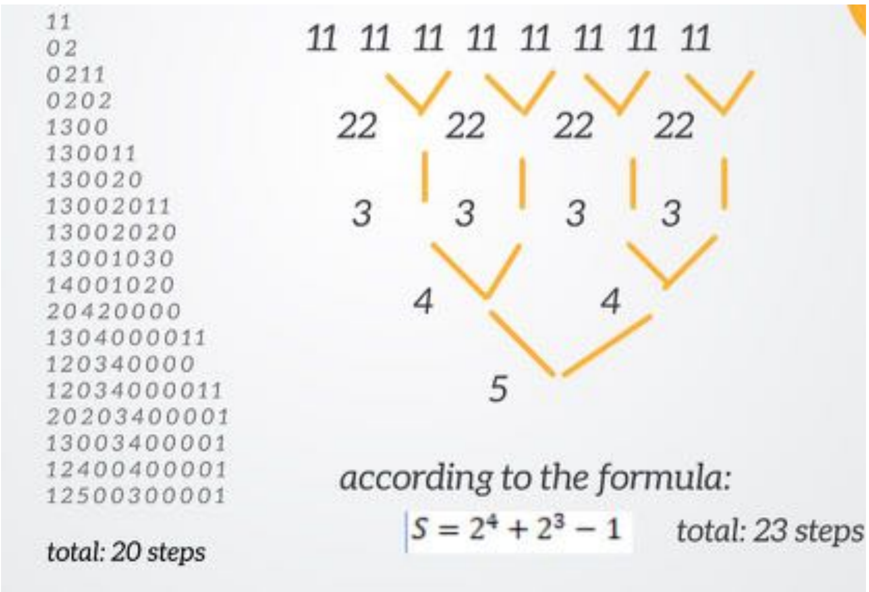
$$2S = 2^1 + 2^2 + 2^3 \ldots + 2^{2016} + 2^{2016}$$

$$\rightarrow \quad S = 2S - S = 2^{2016} + 2^{2016} - 2^{2015} - 1 =$$

$$2^{2016} + 2^{2015}(2-1) - 1$$

$$= 2^{2016} + 2^{2015} - 1$$

Unfortunately, after the joy of thinking that we solved the problem, we had found a contradiction (1).



B. The good idea

After rethinking the problem, we have found the main idea for our subject in order to have the minimum number of steps.

- Each time we find a couple, we will transform it (so that we get the minimum number of steps) (2).
- If we have a choice between 2 couples, it is preferable that we transform the couple with greater numbers.
- If the pairs are only couples of zeroes, then we add two ones again and we do not change them.

Using these rules, we have found the next lines. Each line represents one step.

- 1 1
- 20
- 2011
- 2020
- 3100
- 310011
- 321000

We wanted to discover a rule which could have helped us to find the number of the line and we wanted to find a connection between the numbers on the line and the line itself. At every step, there are two possibilities, either we add 1 1, either we erase $n\ n$ and add $n-1$ and $n+1$. And we found out that ***if we calculate the sum of squares of the numbers on every line, we find an interesting property: the sum increases by 2***.

- If we have a line with some numbers which sum of squares is S and we add 1 1, then the sum of squares on the next line becomes $S+1^2+1^2=S+2$.
- If we have a line with some numbers and also, the numbers $n\ n$, the sum of squares will be $S+n^2+n^2=S+2n^2$ (3). If we erase them and we will add $n-1$ and $n+1$, the sum of squares becomes:

$$\mathrm{S}+(\mathrm{n}-1)^2+(\mathrm{n}+1)^2=\mathrm{S}+\mathrm{n}^2-\ 2\mathrm{n}+1+\mathrm{n}^2+2\mathrm{n}+1=\mathrm{S}+2\mathrm{n}^2+2.$$

The reason why these relations will help us further is, maybe not so obviously, reaching the number of steps. *As we sum up the squares of the numbers from each line, we get a certain sum that divided by 2 will offer us the exact number of steps*, as presented lower:

11
20
2011
2020
3100
310011
321000

1 1	$1^2+1^2=2=2\times 1$
2 0	$2^2+0^2=\ 4=2\times 2$
2 0 1 1	$2^2+0^2+1^2+1^2=6=2\times 3$
2 0 2 0	$2^2+2^2=8=2\times 4$
1 3 0 0	$1^2+3^2=10=2\times 5$
1 3 0 0 1 1	$1^2+1^2+1^2+3^2=12=2\times 6$

C. The last line

In order to calculate the number of steps, we need the numbers on the last line. Lower, we pointed out the lines where the greatest number appears for the first time (4). On one of these lines, there could be either the following:

Rule 1) N, N-2, N-3, . . . , 3, 2 , 1

OR

Rule 2) N, N-2, N-3, . . . , 3, 2, 1, 1

We will prove those rules using induction. For the first case, when N is 3 for example, we see that one of that rules applies. We assume that the rule applies for N and we will prove that the rule applies also for N+1.

I will assume that the following rule applies (5):

N N-2 N-3 N-4 .... 3 2 1

N N-2 N-3 N-4 .... 3 2 1 1 1

N N-2 N-3 N-4 .... 3 2 2 1 0

N N-2 N-3 N-4 .... 5 4 3 3 1 1 0

N N-2 N-3 N-4 .... 5 4 4 2 1 1 0

....

N N-2 N-3 N-3 N-5 ..... 5 4 3 2 1 1 0

N N-2 N-2 N-4 N-5....... 4 3 2 1 1 0

N N-1 N-3 N-4 N-5......... 4 3 2 1 1 0

N-2 IS MISSING

N N-1 N-3 N-4 N-5......... 4 3 2 2 0 0

N N-1 N-3 N-4 N-5 .... 4 3 3 1 0 0

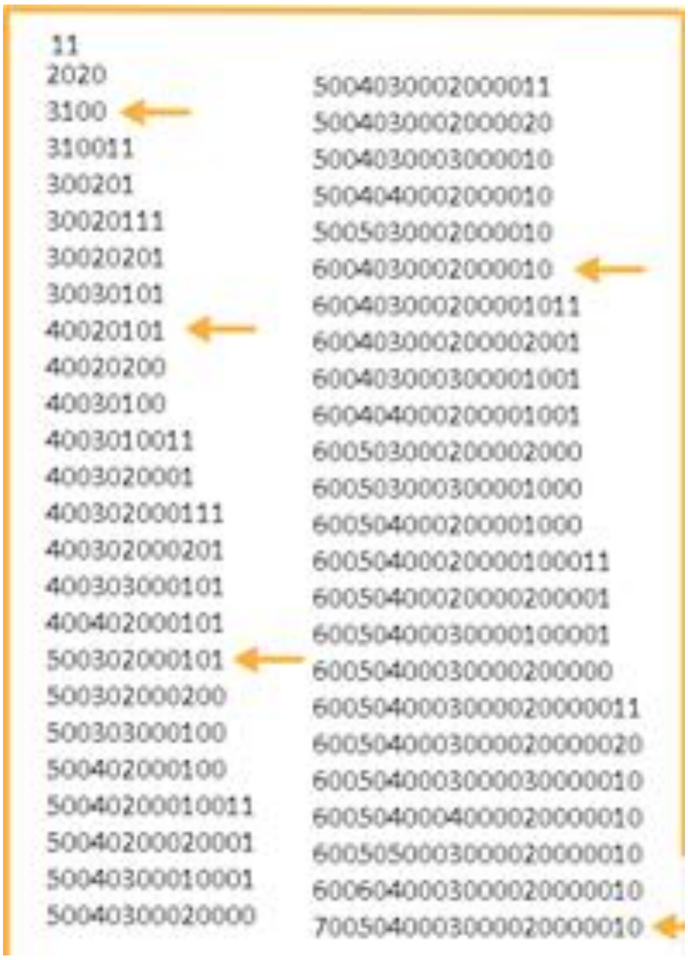
N N-1 N-3 N-4 N-5...... 5 4 4 2 1 0 0

....

N N-1 N-3 N-3 N-5 ...... 5 4 3 2 1 0 0

N N-1 N-2 N-4 N-5 ...... 5 4 3 2 1 0 0

N-3 IS MISSING



Now, every time we will add 1 1, that sequence will repeat, but finally, on the last line another number will miss. For example, on the first blue line N-2 in missing, then on the second one, N-3 is missing. On the next one N-4 will miss, then N-5 and so on until we will reach this line:

N N-1 N-2 N-3 N-4........ 5 4 3 1 1 (2 IS MISSING) (6)

N N-1 N-2 N-3 N-4........ 5 4 3 2 0

N N-1 N-2 N-3 N-4........ 5 4 3 2 0 1 1

N N-1 N-2 N-3 N-4 .... 5 4 3 2 2 0 0

N N-1 N-2 N-3 N-4 .... 5 4 3 3 1 0 0

....

N N-1 N-1 N-3 .... 3 2 1 0 0

N N N-2 N-3 N-4 .... 3 2 1 0 0

N+1 N-1 N-2 N-3 N-4........... 3 2 1 0 0

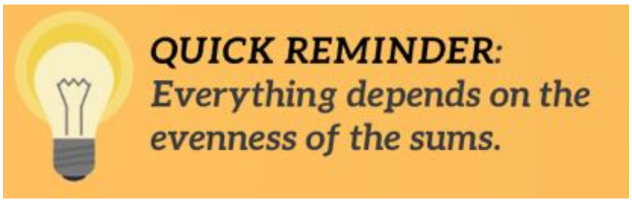
WE DID NOT WRITE ALL THE 0 FROM EVERY LINE AND ALSO ON THE LAST LINE IT COULD BE ONE NUMBER OF 1 OR TWO NUMBERS OF 1, THIS DEPENDS AND THIS IS WHY THERE ARE 2 RULES. WE WILL PROVE BELOW WHICH RULE APPLY.

D. The evenness of the sum

The sum remains even at all times.

We start with 1 1. If we sum them up, we get 2, an even number. Then, the sum of 1 1 couples that we add in order to reach the wanted number is even, and this sum adds to the previous sum (which is also even), resulting in an even sum. The case in which we end up with n n couples ($n + n = 2n$ is even) we will write $n - 1$ and $n + 1$ whose sum is even also ($n - 1 + n + 1 = 2n$).

Coming back, we have to find out when to apply the first rule and when to apply the second one, in order to precisely reach the last line.



Let's take some examples:

40020101

6004030002000010

If the sum is even (7):

A. If the greatest number is even, we won't need another 1, so the first rule applies.
(ex. **6**004030002000010)

B. If the greatest number is odd, we need another 1, so the second rule applies.
(ex. **5**00302000101)

If the sum is odd:

A. If the greatest number is even, we need another 1, so the second rule applies.
(ex. **4**0020101).

B. If the greatest number is odd, we don't need another 1, so the first rule applies
(ex. **7**00504000300002000010).

E. The actual last line

In order to check whether or not there is another 1 on the last line, we have to consider the evenness of the sum, excluding 2017.

Our last line: 2017 2015 2014 2013 … 3 2 1 [1?]

$$\begin{aligned} S &= 2015 + 2014 + 2013 + \ldots + 3 + 2 + 1 \\ &= (2015 + 1) + (2014 + 2) + (2013 + 3) + \ldots + (1009 + 1007) + 1008 \\ &= 2016 \times 1007 + 1008 = 1008 \times (2014 + 1) = 2015 \times 2016 / 2 = 2031120 \end{aligned}$$

The sum is even and 2017 is odd, therefore there are two ones on the last line:

2017 2015 2014 2013 … 3 2 1 **1**

F. We get 2017

The number of steps for the wanted number

$$S = 1^2 + 1^2 + 2^2 + 3^2 + \cdots + 2014^2 + 2015^2 + 2017^2$$

$$S' = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n \times (n+1) \times (2n+1)}{6}$$

Let's prove this through induction. Following the steps of induction proof, we have to verify the formula for n=1. Then, we suppose that the formula is accurate for some $k \geq 1$ and we verify it for $n = k + 1$.

$$1^2 = 1$$

$$\frac{(1+1)(2+1)}{6} = 1$$

For $n = k + 1$

$$S' = 1^2 + 2^2 + 3^2 + \cdots + (k+1-1)^2 + (k+1)^2$$
$$= \frac{k \times (k+1) \times (2k+1)}{6} + (k+1)^2 = \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)]$$
$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6) = \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{n(n+1)(2n+1)}{6}$$

$$S_{2017} = \frac{\frac{2015 \times 2016 \times 4031}{6} + 1 + 2017^2}{2} = 1366608265$$

FINAL THOUGHTS

Our problem is not based on very difficult formulas, having the possibility to be understood by all ages. What would attract the public would be the fact that it is easy to comprehend, even if it has very interesting ideas. This problem is an opened universe even for children, being a problem that becomes more and more interesting as you discover new ideas. What links our problem to the subject of movement? We are playing with the numbers, we make the digits move and we give them sense. Every digit has a purpose. We use endless lines of numbers, but by moving them and putting them in order, not only we simplify our work, but we also make associations and discover new properties.

Take a look at this line, it is meaningless: 00004 10020006003,

but if we move the numbers, we get something interesting: 6 4 3 2 1 00000000000

In mathematics everything is moving. We cannot wait a number to show us a universe. We have to move the numbers in order to find the universe. And our problem let us do that.

Editing Notes

(1) This is not really a contradiction. The example below shows that the “bad idea“ doesn’t give the least number of steps to reach a given number.

(2) This strategy is certainly a good idea, much more efficient than the “bad“ one, as the final result shows, and one can conjecture that it is the best possible.
However, it is not proved that it gives the minimal number of steps.

(3) This time, the sum S doesn’t include the squares of the pair n n to be deleted.

(4) In the table below, two lines are missing between 11 and 2020 : 20 and 2011.

(5) Here the first sequence ends with 3 2 1, according to “rule 1”, and we find later on “blue“ lines (lines without pair n n other than 1 1) ending 3 2 1 1 and 3 2 1 alternately.
If we begin with “rule 2”, the scheme is the same except that these last terms will appear in the opposite order.

(6) The line here could as well end with a single 1, depending on the parity of N (here, with a pair of 1, we have an odd number of alternances between “N-1 is missing“ and “2 is missing“, so N has to be even).

(7) Here, *the sum* refers to the sum of the numbers except for the largest one, and taking in account only one 1, that is $(n-2)+(n-3)+\cdots+2+1$ if the largest number is n.