Tiling design

Year 2018– 2019

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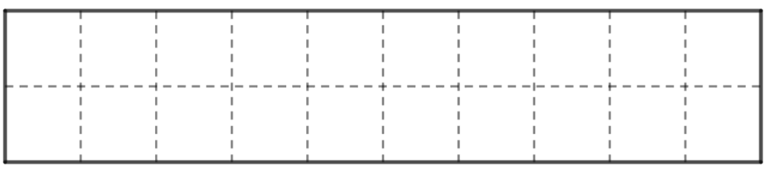
1. Presentation of the research topic

A tiling design is a way of arranging various plane shapes (tiles) so that they completely cover a given area without overlapping and with no gaps. Despite their playful appearance, tiling problems have relevance to architecture and decorative arts, computer graphics, but also to Statistical Mechanics, where tilings can be used as models for molecule arrangements on a lattice. For example, if one considers a system in which a lattice is covered by monomers (basically a single molecule, modelled by 1×1 squares), dimers (a bond of two structurally similar monomers, modelled by dominos), trimers (a combination of three monomers, modelled by trominos) etc, then the thermodynamical properties of the system can be derived from the number of arrangements that can be found at its steady states (states with zero energy).

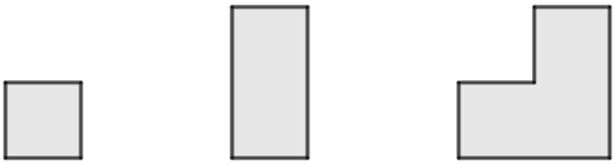
Our work focuses on counting the number of different tiling designs that can be obtained on a 2×10 or a 3×10 lattice, using three types of tiles: 1x1 squares, dominos and L-shaped trominos. To this end, we have derived appropriate recurrence relations, based on which we were able to find the number of all possible tiling designs. For all items in the research proposal, we have provided detailed combinatorial solutions, with suggestive graphical representations. Similar work can be done for more general lattices, such as chessboards or non-rectangular shaped lattices.

2. The topic

A 2×10 rectangular wall is going to be covered with tiles.



We have at our disposal three types of tiles, as shown in the figure below.



The tiles are unlimited and can be rotated in any way before being assembled.

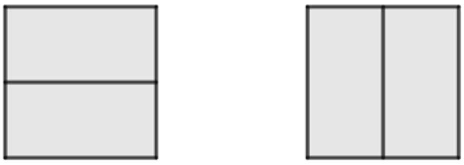
a) *In how many ways can we cover the wall only with domino tiles?*
b) *In how many ways can we cover the wall if the available tiles are of two types: type 1×1 and type 2×1.*
c) *What happens if we have all three types of tile available as drawn above?*
d) *In how many ways can we cover a 3×10 rectangular wall only with domino tiles?*

3. The Solution

a) Let us solve the problem from *a)* in the $2\times n$ case (2 lines and n columns). Let us consider a_n as the number of ways in which we can completely cover, without overlapping, a $2\times n$ wall with tiles 1×2.

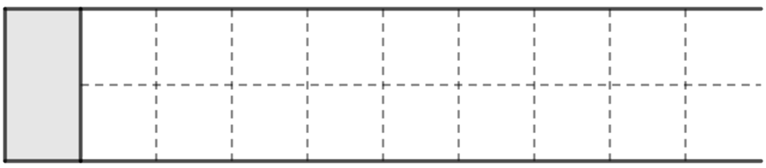
We consider $a_0=1$. Because a 2×1 wall can be covered in only one way with a piece of tile 1×2 vertically placed as in the figure below, it follows that $a_1=1$.

For a 2×2 wall we have two possibilities, two 1×2 tiles, horizontally placed one above the other and two tiles 1×2, vertically placed:

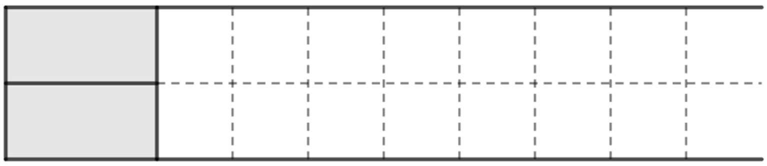


So, $a_2=2$.

Let`s see now how can a $2\times n,\ n\geq3$, wall be covered.
– I. If we complete the first column with a 1×2 vertical tile, then the rest of the wall can be covered in a_{n-1} ways.



– II. If we cover the first two columns with two 1×2 tiles horizontally arranged, one under the other, then the rest of the wall will be covered in a_{n-2} ways.



Because these are the only possibilities to begin the covering, it results that

$$a_n = a_{n-1} + a_{n-2}, \text{ for any natural number } n,\ n \ge 3.$$

We observe that the formula is also correct for $n \ge 2$, with the convention that $a_0 = 1$.
It results that $a_2 = a_1 + a_0 = 2$, $a_3 = a_2 + a_1 = 3$, $a_4 = a_3 + a_2 = 5$, ..., $a_{10} = a_9 + a_8 = 89$.
In conclusion, the number of possibilities in which we can cover a 2×10 wall with 1×2 tiles is $a_{10} = 89$.

Observation. The formulas

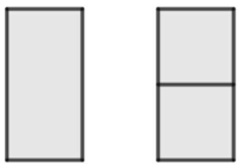
$$\begin{cases} a_0 = a_1 = 1 \\ a_n = a_{n-1} + a_{n-2} \end{cases}$$

show that $(a_n)_{n\ge 0}$ is, in fact, the well-known Fibonacci sequence. It is known that (1)

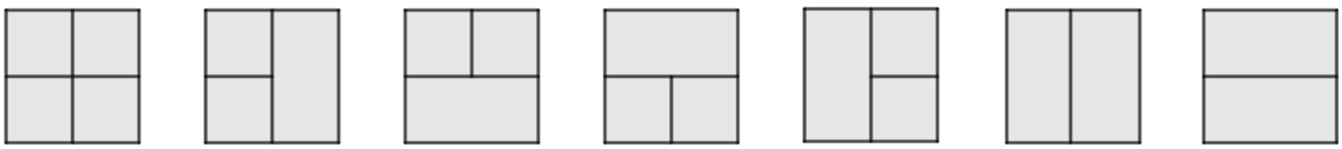
$$a_{n-1} = F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \text{ for any natural number } n \ge 1.$$

***b)* Solution 1.**

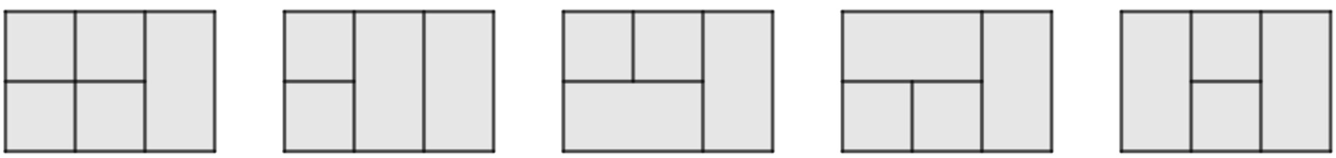
Let us consider b_n be the number of coverings of a $2\times n$ wall with square 1×1 tiles or 1×2 dominoes.
We consider $b_0 = 1$. We observe that:
$b_1 = 2$

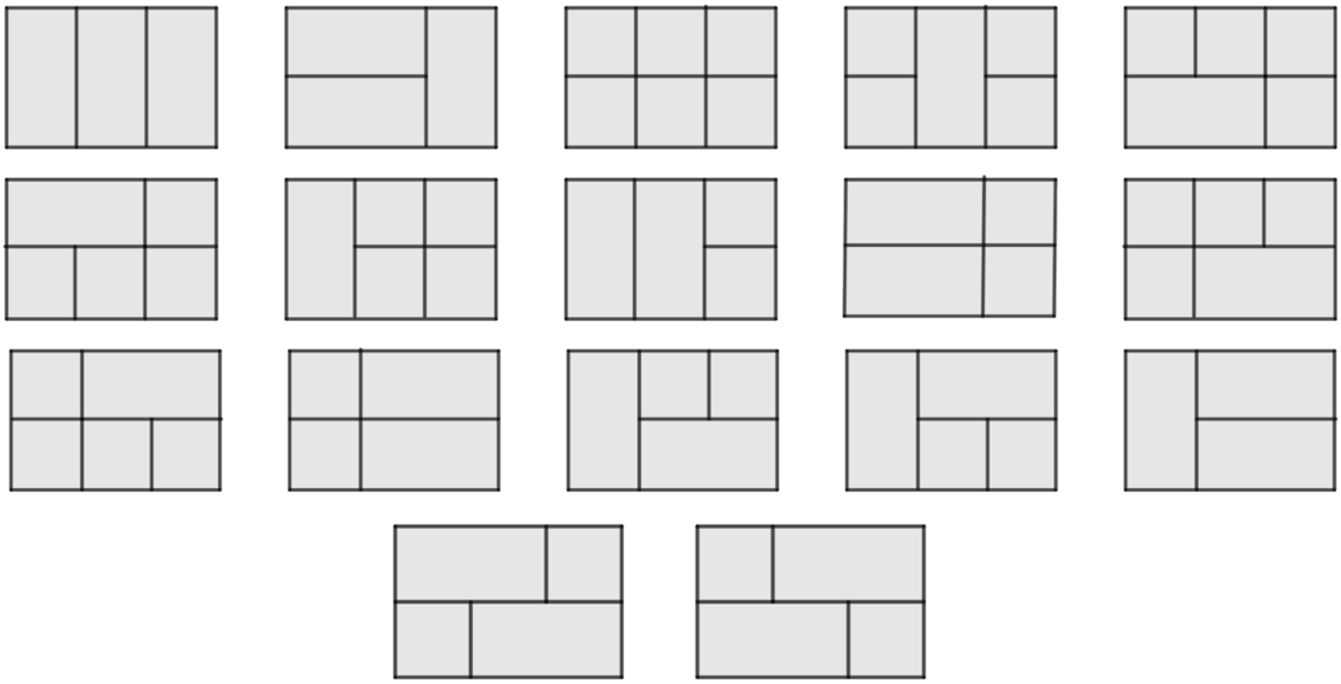


$b_2 = 7$



$b_3 = 22$





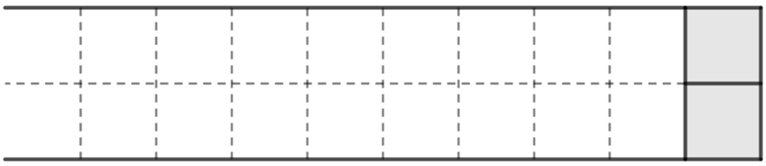
To find the values of b_n for $n \geq 3$, we will use the next result.

Theorem 1. *For every* $n \geq 3,$ *we have*

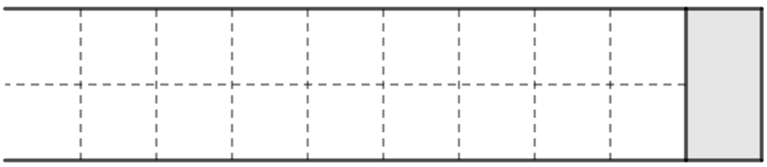
$$b_n = 3b_{n-1} + b_{n-2} - b_{n-3}.$$

Proof.

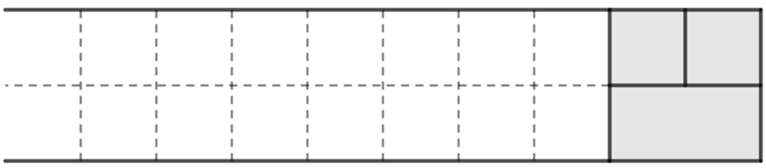
There are b_{n-1} coverings of a $2 \times n$ wall which finish with two 1×1 square tiles in the n^{th} column:



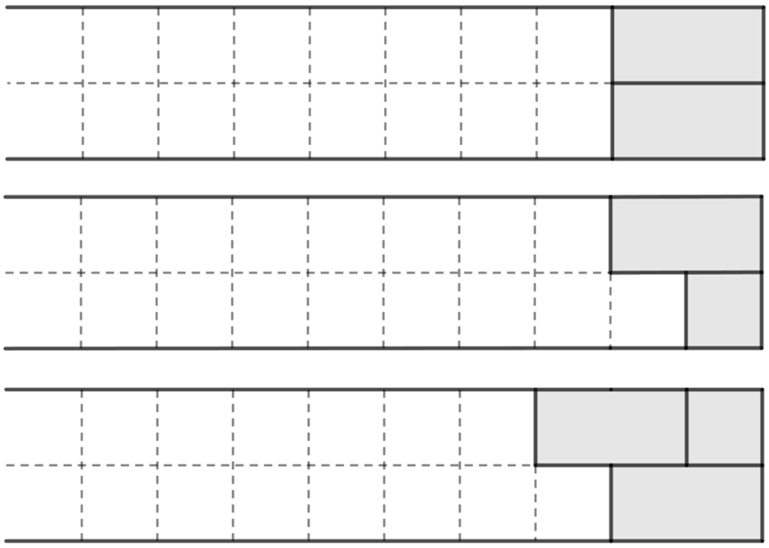
There are b_{n-1} coverings which finish with a 1×2 domino tile arranged vertically in the n^{th} column:



There are b_{n-2} coverings which finish with two 1×1 square tiles arranged on the upper row and a horizontal 1×2 domino tile on the lower row, in the $(n-1)^{th}$ and n^{th} column:

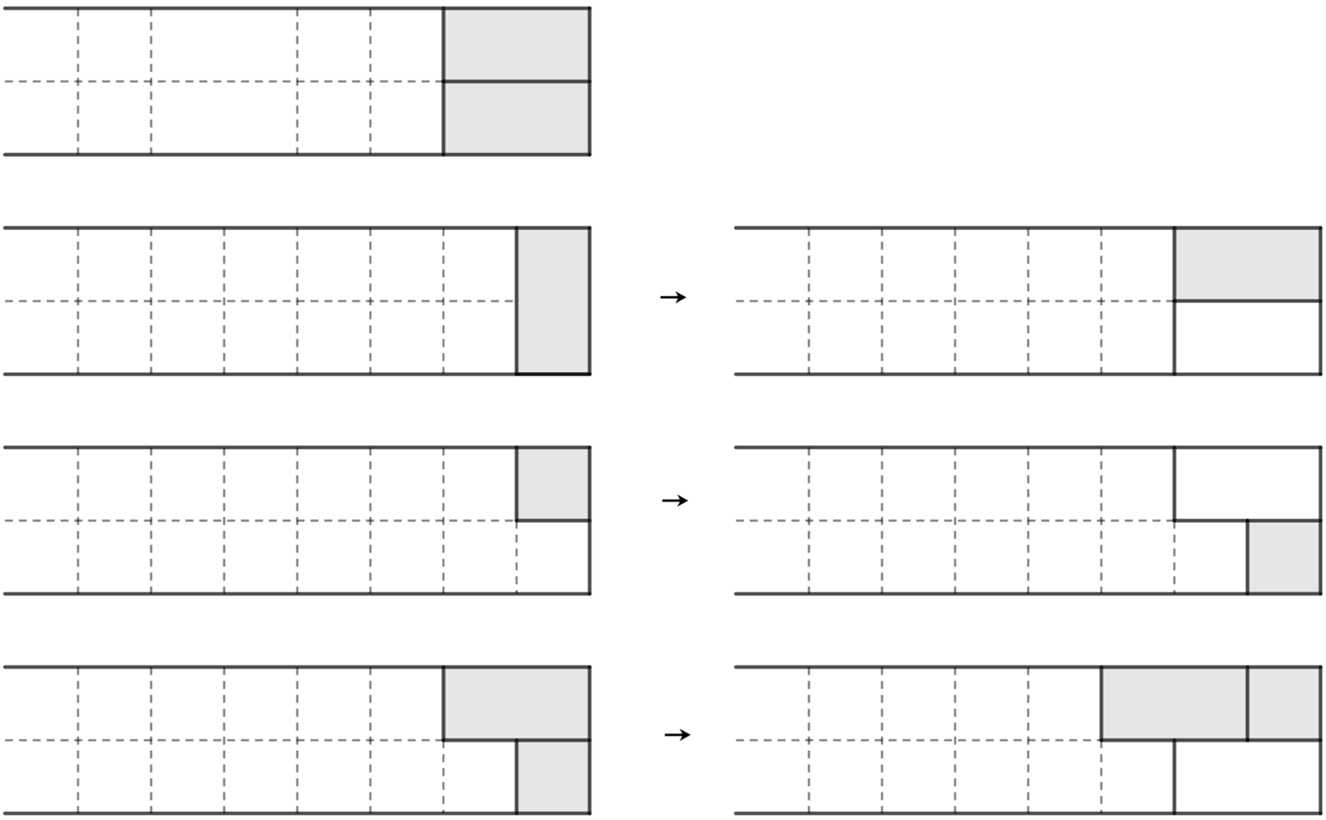


The remaining coverings of a $2\times n$ wall can finish in the following ways:



Let us denote the above coverings with $(*)$. We will show that the number of $(*)$ coverings is $b_{n-1}-b_{n-3}$.

A covering of $2\times(n-1)$ can be finished in the following ways:



Each of the last three coverings of a $2\times(n-1)$ wall can be transformed, uniquely, in a covering of a $2\times n$ wall of $(*)$ form (2). So, the number of coverings of the $(*)$ form is $b_{n-1}-b_{n-3}$.

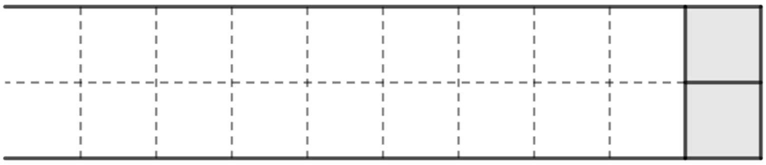
■

Using Theorem 1, we get $b_0 = 1$, $b_1 = 2$, $b_2 = 7$, $b_3 = 3b_2 + b_1 - b_0 = 21 + 2 - 1 = 22$ (we have also obtained this result by direct counting), $b_4 = 3b_3 + b_2 - b_1 = 66 + 7 - 2 = 71$, $b_5 = 3b_4 + b_3 - b_2 = 213 + 22 - 7 = 228$, …, $b_{10} = 3b_9 + b_8 - b_7 = 78243$.

In conclusion, the number of possibilities in which we can cover a 2×10 wall with 1×2 domino tiles and 1×1 square tiles is $b_{10} = 78\,243$.

***b)* Solution 2.**

- Let x_n be the number of coverings of a $2\times n$ wall ended in this way:



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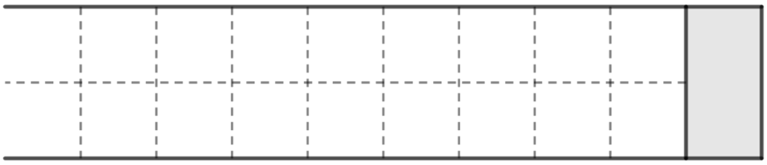

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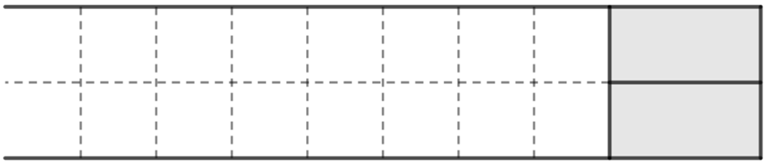
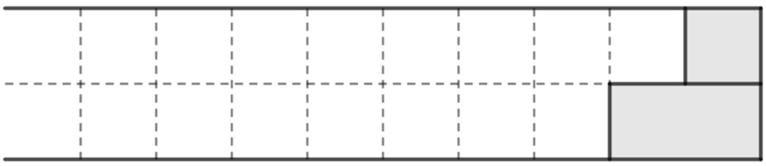
- Let y_n be the number of coverings of a $2\times n$ wall ended in this way:



- Let z_n be the number of coverings of a $2\times n$ wall ended in this way:



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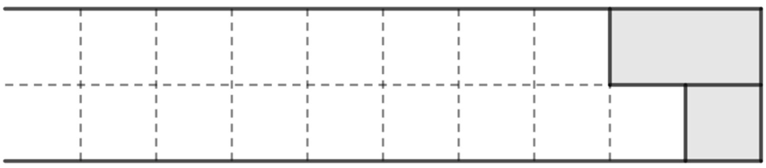

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So, we have (3)

$$(1)\quad \begin{cases} b_n = x_n + y_n + z_n \text{, for any } n \ge 1 \\ x_n = 2b_{n-1} \text{, for any } n \ge 2 \\ y_n = b_{n-2} \text{, for any } n \ge 3 \\ z_n = 2b_{n-2} + z_{n-1} \text{, for any } n \ge 3 \end{cases}.$$

Using these formulas, we can find the value of b_{10}:

$$x_1 = 2,\ y_1 = 0,\ z_1 = 0 \ \Rightarrow\ b_1 = 2+0+0 = 2$$
$$x_2 = 2\cdot 2 = 4,\ y_2 = 1,\ z_2 = 2 \ \Rightarrow\ b_2 = 4+1+2 = 7$$
$$x_3 = 2\cdot 7 = 14,\ y_3 = 2,\ z_3 = 2\cdot 2+2 = 6 \ \Rightarrow\ b_3 = 14+2+6 = 22$$
$$x_4 = 2\cdot 22 = 44,\ y_4 = 7,\ z_4 = 2\cdot 7+6 = 20 \ \Rightarrow\ b_4 = 44+7+20 = 71$$
$$x_5 = 2\cdot 71 = 142,\ y_5 = 22,\ z_5 = 2\cdot 22+20 = 64 \ \Rightarrow\ b_5 = 228$$
$$x_6 = 2\cdot 228 = 456,\ y_6 = 71,\ z_6 = 2\cdot 71+64 = 206 \ \Rightarrow\ b_6 = 733$$
$$x_7 = 2\cdot 733 = 1466,\ y_7 = 228,\ z_7 = 2\cdot 228+206 = 662 \ \Rightarrow\ b_7 = 2356$$
$$x_8 = 2\cdot 2356 = 4712,\ y_8 = 733,\ z_8 = 2\cdot 733+662 = 2128 \ \Rightarrow\ b_8 = 7573$$
$$x_9 = 2\cdot 7573 = 15146,\ y_9 = 2356,\ z_9 = 2\cdot 2356+2128 = 6840 \ \Rightarrow\ b_9 = 24342$$
$$x_{10} = 2\cdot 24342 = 48684,\ y_{10} = 7573,\ z_{10} = 2\cdot 7573+6840 = 21986 \ \Rightarrow\ b_{10} = 78243$$

In this way, to find b_{10} we need to compute all the values of x_n, y_n, z_n, for $n=1$ to $n=10$. It is natural to wonder if it is not possible to eliminate x_n, y_n, z_n, and find a recurrence formula for the sequences $(b_n)_{n\ge 1}$. We will show that this is possible. From (1) it follows that

$$b_n = x_n + y_n + z_n = 2b_{n-1} + 3b_{n-2} + z_{n-1} \text{, for any } n \ge 3,$$

and therefore

$$b_{n+1} = 2b_n + 3b_{n-1} + z_n \text{, for any } n \ge 2.$$

By subtracting the last two equations we get

$$b_{n+1} - b_n = 2b_n + b_{n-1} - 3b_{n-2} + (z_n - z_{n-1}) = 2b_n + b_{n-1} - 3b_{n-2} + 2b_{n-2} \text{, for any } n \ge 3,$$

thus

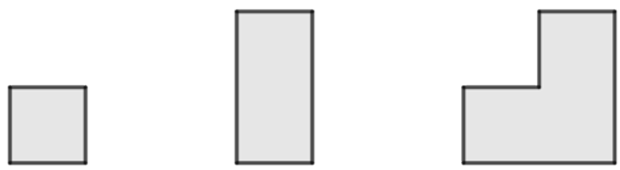
$$b_{n+1} = 3b_n + b_{n-1} - b_{n-2} \text{, for any } n \ge 3,$$

or

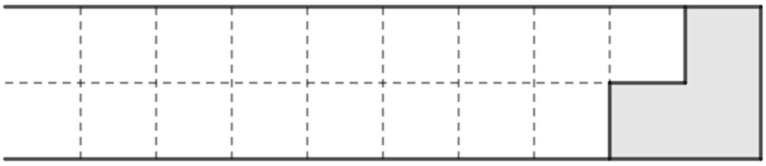
$$b_n = 3b_{n-1} + b_{n-2} - b_{n-3} \text{, for any } n \ge 4,$$

which is the formula of Theorem 1.

c) Let c_n be the number of ways we can tile a $2\times n$ wall with the following shapes:



Let x_n be the number of coverings that can end the following ways:



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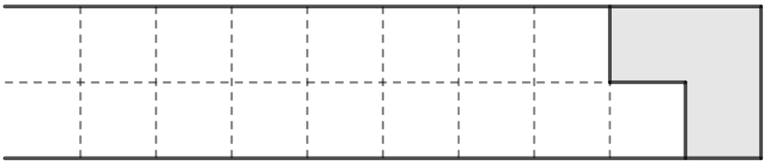

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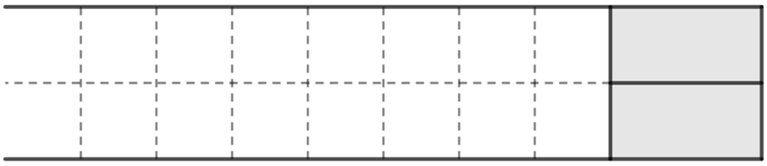

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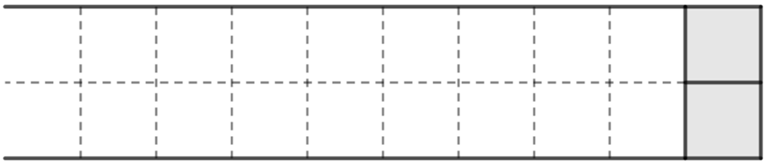
Let y_n be the number of coverings that can end the following ways:



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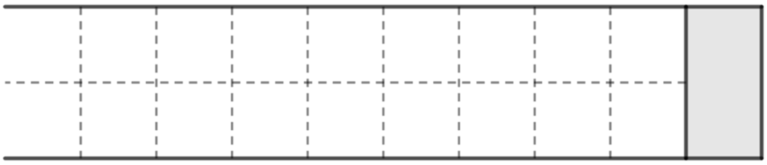
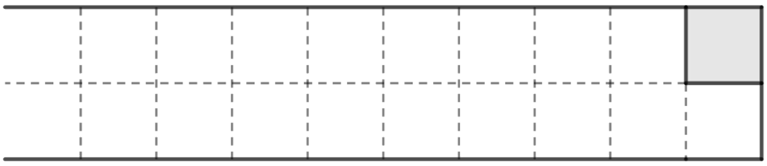
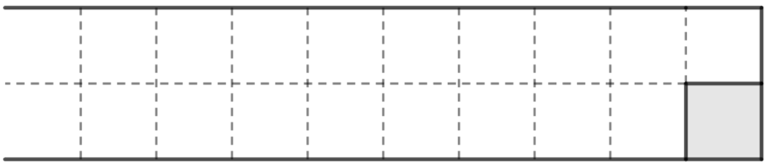


Let z_n be the number of coverings whose ends don't fit anywhere above:





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but aren't counted in y_n.

Clearly, we have (4):

$$(2) \qquad \begin{cases} c_n = x_n + y_n + z_n, \text{ for any } n \geq 1 \\ x_n = 3c_{n-2} + z_{n-1}, \text{ for any } n \geq 3 \\ y_n = 2c_{n-1}, \text{ for any } n \geq 2 \\ z_n = 4c_{n-2} + z_{n-1} \text{ for any } n \geq 3 \end{cases}.$$

Using these formulas, we can find the value of c_{10}:

$$x_1 = 0,\ y_1 = 2,\ z_1 = 0 \ \Rightarrow\ c_1 = 2$$
$$x_2 = 3,\ y_2 = 4,\ z_2 = 4 \ \Rightarrow\ c_2 = 11$$
$$x_3 = 10,\ y_3 = 22,\ z_3 = 12 \ \Rightarrow\ c_3 = 44$$
$$x_4 = 45,\ y_4 = 88,\ z_4 = 56 \ \Rightarrow\ c_4 = 189$$
$$x_5 = 188,\ y_5 = 378,\ z_5 = 232 \ \Rightarrow\ c_5 = 798$$
$$x_6 = 799,\ y_6 = 1596,\ z_6 = 988 \ \Rightarrow\ c_6 = 3383$$
$$x_7 = 3382,\ y_7 = 6766,\ z_7 = 4180 \ \Rightarrow\ c_7 = 14328$$
$$x_8 = 14329,\ y_8 = 28656,\ z_8 = 17712 \ \Rightarrow\ c_8 = 60697$$
$$x_9 = 60696,\ y_9 = 121394,\ z_9 = 75024 \ \Rightarrow\ c_9 = 257114$$
$$x_{10} = 257115,\ y_{10} = 514228,\ z_{10} = 317812 \ \Rightarrow\ c_{10} = 1089155.$$

From (2) we can deduce a recurrence formula for $(c_n)_{n\geq 1}$. Indeed, we have:

$$c_n = x_n + y_n + z_n = 3c_{n-2} + z_{n-1} + 2c_{n-1} + 4c_{n-2} + z_{n-1} = 2c_{n-1} + 7c_{n-2} + 2z_{n-1}, \text{ for any } n \geq 3,$$

thus

$$c_{n+1} - c_n = 2c_n + 5c_{n-1} - 7c_{n-2} + 2(z_n - z_{n-1}) = 2c_n + 5c_{n-1} - 7c_{n-2} + 8c_{n-2} = 2c_n + 5c_{n-1} + c_{n-2}, \text{ for any } n \geq 3$$

,

so

$$c_{n+1} = 3c_n + 5c_{n-1} + c_{n-2}, \text{ for any } n \geq 3,$$

or

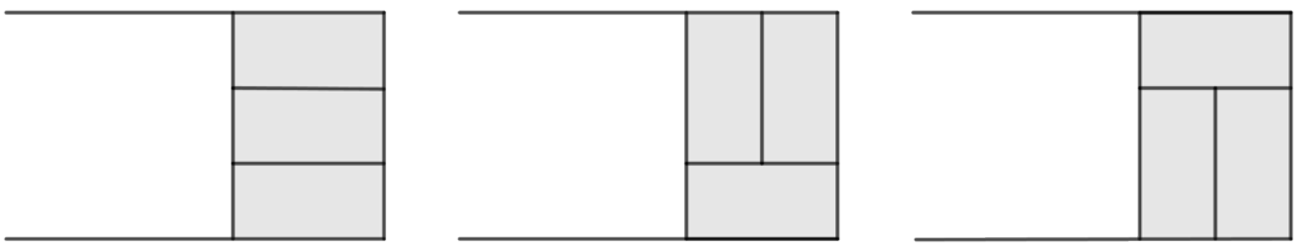
$$c_n = 3c_{n-1} + 5c_{n-2} + c_{n-3}, \text{ for any } n \geq 4,$$

which, knowing that $c_1 = 2,\ c_2 = 11$ and $c_3 = 44$, gives us $c_{10} = 1089155$.

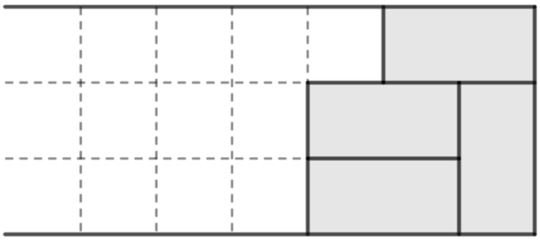
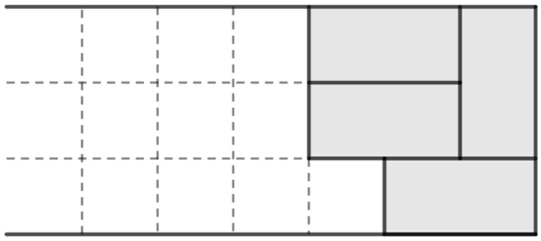
d) Solution 1

Let d_n be the number of ways we can cover a $3 \times n$ rectangular wall with 2×1 domino tiles.

We denote with x_n the number of coverings which finish like this:



We denote with y_n the number of coverings which finish like this:

We have

$$(3)\qquad \begin{cases} d_n = x_n + y_n, \text{ for any } n \geq 1 \\ x_n = 3 \cdot d_{n-2}, \text{ for any } n \geq 3 \\ y_n = 2d_{n-4} + y_{n-2}, \text{ for any } n \geq 5 \end{cases}$$

Using (3), we find

$$x_1 = 0,\ y_1 = 0 \ \Rightarrow\ d_1 = 0$$
$$x_2 = 3,\ y_2 = 0 \ \Rightarrow\ d_2 = 3$$
$$x_3 = 0,\ y_3 = 0 \ \Rightarrow\ d_3 = 0$$
$$x_4 = 9,\ y_4 = 2 \ \Rightarrow\ d_4 = 11$$
$$x_5 = 0,\ y_5 = 0 \ \Rightarrow\ d_5 = 0$$
$$x_6 = 33,\ y_6 = 8 \ \Rightarrow\ d_6 = 41$$
$$x_7 = 0,\ y_7 = 0 \ \Rightarrow\ d_7 = 0$$
$$x_8 = 123,\ y_8 = 30 \ \Rightarrow\ d_8 = 153$$
$$x_9 = 0,\ y_9 = 0 \ \Rightarrow\ d_9 = 0$$
$$x_{10} = 459,\ y_{10} = 112 \ \Rightarrow\ d_{10} = 571.$$

Let us find now a recurrence formula for the sequence $(d_n)_{n\geq 1}$.

From (3), we get:

$$d_n = x_n + y_n = 3d_{n-2} + 2d_{n-4} + y_{n-2}, \text{ for any } n \geq 5,$$

thus

$$d_{n+2} = 3d_n + 2d_{n-2} + y_n, \text{ for any } n \geq 3,$$

therefore

$$d_{n+2} - d_n = 3d_n + 2d_{n-2} + y_n - 3d_{n-2} - 2d_{n-4} - y_{n-2} = 3d_n - d_{n-2} - 2d_{n-4} + (y_n - y_{n-2}) =$$
$$3d_n - d_{n-2} - 2d_{n-4} + 2d_{n-4} = 3d_n - d_{n-2},$$

so

$$d_{n+2} = 4d_n - d_{n-2}, \text{ for any } n \geq 3.$$

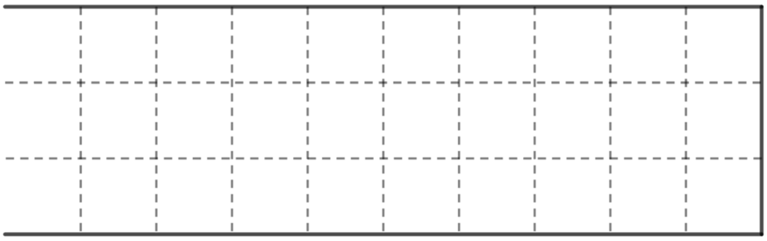
Finally, we have

$$d_n = 4d_{n-2} - d_{n-4}, \text{ for any } n \geq 5.$$

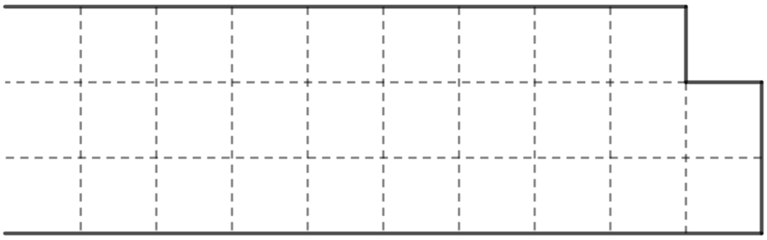
With this formula, and with $d_1 = d_3 = 0$, $d_2 = 3,\ d_4 = 11$, we can compute d_n, for any natural number *n*. This way, we get $d_{10} = 571$.

d) **Solution 2.**

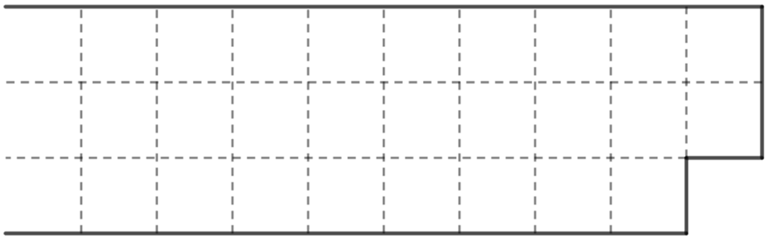
Let d_n be the number of ways we can put the domino tiles, so we can have a complete $3\times n$ wall.



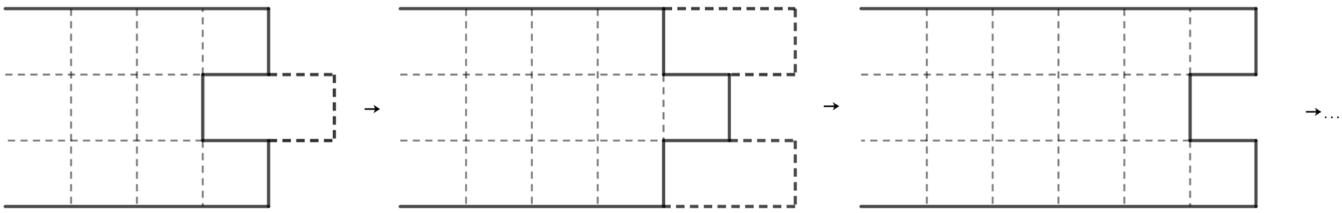
Let x_n be the number of ways we can arrange the domino tiles, so we have a $3\times n$ wall, but with the top right corner missing.



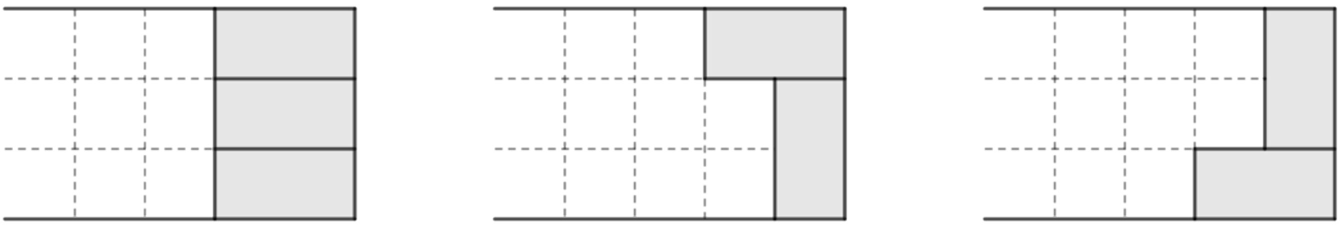
Let y_n be the number of ways we can arrange the domino tiles, so we have a $3\times n$ wall, but with the bottom right corner missing.



These are all the possible ways the last column can end, any other possibility leading to a configuration with an infinite number of domino tiles (5).



All the possibilities from which we can obtain a d_n type of tiling based on pre-calculations are:

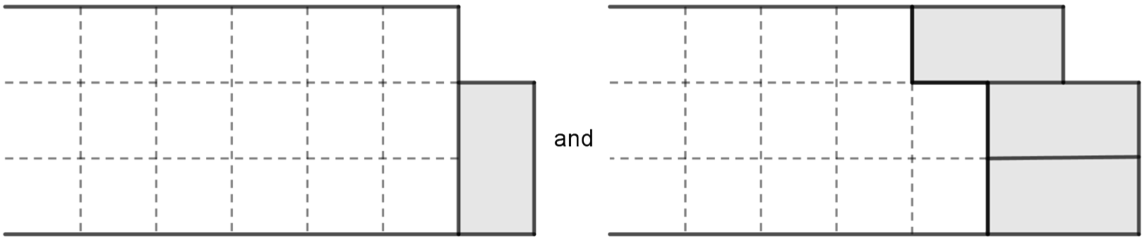


So,

$$d_n = d_{n-2} + x_{n-1} + y_{n-1}, \text{ for any even } n \geq 4 .$$

Observation. We only have solutions for an even n in d_n, respectively an odd n in x_n and y_n.

All the possibilities in which we can obtain a x_n type of tiling based on pre-calculations (and not counted twice) are:



So,

$$x_n = d_{n-1} + x_{n-2}, \text{ for any odd } n \geq 3 .$$

Observation: $x_n = y_n$, for any odd n**,** because every arrangement counted in x_n can be associated with the same configuration from y_n, but flipped horizontally.

We have the final formulas:

$$(4) \qquad \begin{cases} d_n = d_{n-2} + 2x_{n-1}, \text{ for any even } n \geq 4 \\ x_n = d_{n-1} + x_{n-2}, \text{ for any odd } n \geq 3 \end{cases},$$

with $d_1 = 0$, $d_2 = 3$, $x_1 = 1$.

From the first equation of (4), we get

$$d_{n+1} - d_{n-1} = d_{n-1} + 2x_n - d_{n-3} - 2x_{n-2} = d_{n-1} - d_{n-3} + 2(x_n - x_{n-2}) = d_{n-1} - d_{n-3} + 2d_{n-1} = 3d_{n-1} - d_{n-3},$$

thus

$$d_{n+1} = 4d_{n-1} - d_{n-3}, \text{ for any } n \geq 3,$$

or

$$d_n = 4d_{n-2} - d_{n-4}, \text{ for any } n \geq 5,$$

which is the same relation as in Solution 1.

The answer is $d_{10} = 571$.

4. Conclusion

We have answered all the questions in the research proposal, by providing detailed solutions and drawing suggestive diagrams. For some of the questions, we have given two different solutions. The solutions are based on the construction of relevant recurrence relations for the number of tiling designs. The recurrences we have derived in this paper are of orders two (for question (a)), three (for questions (b) and (c)) and four (for question (d)). We can see that, the more different types of tiles are used in the pattern design, the higher is the order of the recurrence relation. We expect that, for more complicated lattice shapes, the recurrence relation will have even higher orders, and their derivation will not be easy at all.

Editing notes

(1) A proof of this result can be found for example in Wikipedia.

(2) These transformations are shown in the figure above: one or two tiles are moved and a domino is added.

(3) The formula for z_n is justified as follows: we have two cases depending on whether the white square on line $n-1$ is covered by a 1×1-tile or by a domino. In the first case, the $2 \times n$ covering is obtained from a $2 \times (n-2)$ covering by adding two single squares on one row and one domino on the other (two possibilities); in the other case it is obtained from a covering of type z_{n-1} by moving the isolated gray square of column $n-1$ to column n on the other row, and adding a domino (one possibility).

(4) Note that for a covering of type z_n, the white square of column n must be covered by either a domino or a tromino, since the case of two 1×1-tiles on the n-th column is counted in y_n.
Coverings of type x_n ending with two dominos or a tromino and a 1×1-tile are obtained by completing a $2 \times (n-2)$ covering (3 possibilities for each $2 \times (n-2)$ covering); the others ones end with a tromino but the remaining square on column $n-1$ is covered by a domino or a tromino: they correspond to a covering of type z_{n-1} by replacing the last tromino by a 1×1-tile. The formula $x_n = 3c_{n-2} + z_{n-1}$ follows.
A similar argument shows that there are 4 ways to extend a $2 \times (n-2)$ covering into a covering of type z_n; the coverings of type z_n which cannot be obtained in this way (i.e.. which cannot be cut after column $n-2$) end with a domino on the row opposite the gray square, and the square next to the grey square on the same row is covered by a wider tile: they correspond to a covering of type z_{n-1} by removing the domino and moving the gray 1×1-tile. So, we get $z_n = 4c_{n-2} + z_{n-1}$.

(5) In fact, there is also the possibility of a single square at the top or bottom of the last column. But this has no consequence, since these configurations are not found in the proof below.