

Solar System

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Introduction

Our research tries to answer the following question: What happens of each planet in the Solar System if the Sun is removed?

Our result

We found that:

1. After removing the sun, each planet tends to follow the direction of its instantaneous velocity vector.
2. The planets reshape their trajectory in time, due to the gravitational attraction forces between them.

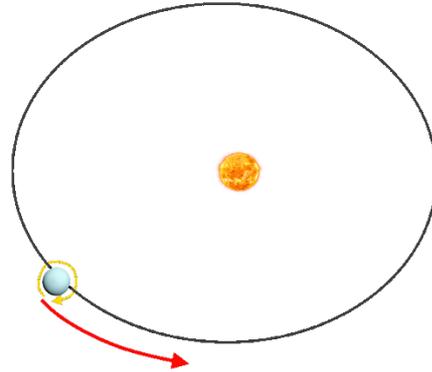
Hypotheses

Some well-known facts are that:

A planet is a massive object orbiting a star.

In the Solar System, there are 8 planets: the first four (Mercury, Venus, Earth, Mars) are terrestrial, while the last four giants (Jupiter, Saturn, Uranus, Neptune) are made of gas.

Each planet performs 2 types of rotations: on its own axis and around the Sun. We will focus on the rotation around the Sun. Our reasoning is that, if we know the rules that celestial bodies follow in their actual motion, then we can apply them to an imaginary system, which meets the requirements of our subject. If we do that, we can also create our own solar system. In the next part, we will consider planets as moving points having their own masses.



Rotation around the Sun (geometry-based model)

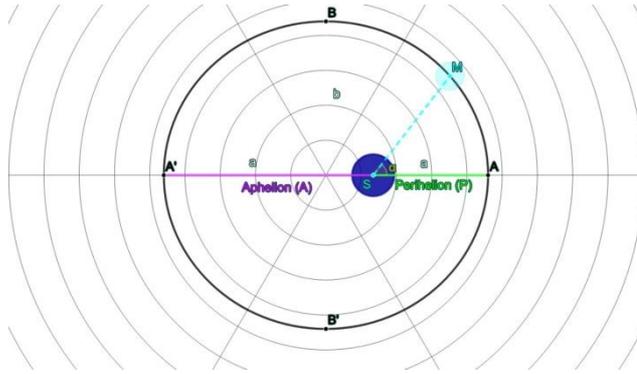
In this part we introduce a new hypothesis: let's think of an imaginary Solar System that follows rules simpler than in reality. We consider a bidimensional grid system where *the Sun is a fixed object* at coordinates (0,0) and the only interaction forces are those between the Sun and each planet, individually. We also assume that *the trajectory described by each planet is a perfect ellipse and the angular speed of each planet is constant*. Having a constant angular speed allows us to know the positions of planets at a certain moment in time. **[1]** In this way, we can create an ideal geometric model of the planet motion.

First of all, we need some information about the ellipse. Starting from the definition (the geometric place of all points that fulfill the property: the sum of distances of each point from two fixed points, called foci, is constant). We will use the common notation for the ellipse parameters: a (major axis), b (minor axis), and c (focal length), along with the equality $b^2 = a^2 - c^2$ (1). We also remind the well-known ellipse equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

Now, we return to the study of the planet motion.

According to the first law of Kepler, a planet revolves around its star in an elliptical orbit that has its star in one of its foci. We managed to find an equation that describes a possible (expected) orbit trajectory of the planet.



We call:

- Aphelion (\mathcal{A}) - the furthest point from the star that the planet reaches on its trajectory
- Perihelion (\mathcal{P}) - the closest point to the star that the planet reaches on its trajectory

Given the aphelion and perihelion distances, we will try to find the values a, b, c , which describe the ellipse form of the trajectory.

We find:

$$\mathcal{A} = a + c, [2]$$

$$\mathcal{P} = a - c,$$

From where

$$\mathcal{A} + \mathcal{P} = 2a \Rightarrow a = \frac{\mathcal{A} + \mathcal{P}}{2} \quad (1.a)$$

$$\mathcal{A} - \mathcal{P} = 2c \Rightarrow c = \frac{\mathcal{A} - \mathcal{P}}{2} \quad (1.b)$$

From the notation we made above(1), we have:

$$\begin{aligned} c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2 &= \left(\frac{\mathcal{A} + \mathcal{P}}{2}\right)^2 - \left(\frac{\mathcal{A} - \mathcal{P}}{2}\right)^2 = \frac{(\mathcal{A} + \mathcal{P})^2 - (\mathcal{A} - \mathcal{P})^2}{4} = \\ &= \frac{\mathcal{A}^2 + 2\mathcal{A}\mathcal{P} + \mathcal{P}^2 - (\mathcal{A}^2 - 2\mathcal{A}\mathcal{P} + \mathcal{P}^2)}{4} = \frac{4\mathcal{A}\mathcal{P}}{4} = \mathcal{A}\mathcal{P} \\ \Rightarrow b &= \sqrt{\mathcal{A}\mathcal{P}} \quad (1.c) \end{aligned}$$

The next step is writing the equation of motion in terms of time. We will take $\alpha(t)$ the angular motion of the planet and $x(t), y(t)$ are the object's position at the moment t .

We start with finding $x(t)$ and $y(t)$ equation forms for an ellipse of axes a, b and center in $(0,0)$ having the known parametric equation:

$$x(t) = a \cos(t)$$

$$y(t) = b \sin(t)$$

Now, assuming that every planet moves faster or slower than each other, according to the hypothesis, we can choose $\alpha(t) = k \cdot t, k \in \mathbb{R}^*$ a linear function in terms of time.

We can write the parametric equation as

$$x(t) = a \cos(\alpha(t))$$

$$y(t) = b \sin(\alpha(t))$$

This equation describes an ellipse for $t \in \left[0, \frac{2\pi}{k}\right]$, which means the greater k is, the faster a cycle is described by the moving point (in our case, a planet) following this equation.

This is the equation of an ellipse with center in the origin of the coordinates system. Considering the fact that the ellipse can be placed anywhere in space, we get the result:

$$x(t) = x_0 + a \cos(\alpha(t)) \quad [F1]$$

$$y(t) = y_0 + b \sin(\alpha(t)) \quad [F2]$$

Particularly, on the basis of the first Kepler's law mentioned at the beginning of this part, we succeeded to find a way to mathematically represent a planet orbit prototype:

$$x(t) = -c + a \cos(\alpha(t)) \quad (2. a)$$

$$y(t) = b \sin(\alpha(t)) \quad (2. b)$$

Note that $\alpha(t)$ is the angle relative to the center of the ellipse $(-c, 0)$, not to the Sun, located at $(0, 0)$.

Replacing a, b, c with the real values computed above in (1. a) – (1. c) as functions of the given distances of Aphelion and Perihelion, we get the orbit corresponding to each planet in the Solar System.

Now, as we observed how planets are moving, we are prepared to study the system without Sun. We think that the planets will move rectilinearly according to the velocity vector tangent to the elliptical trajectory in the moment in which the Sun disappears. The idea is plausible, since, except for the forces of gravitational attraction between bodies, speed is the most important element of the motion. **3**

In order to check the situation, we have to study the tangent to a curve in a given point. That line describes the possible path followed by the planet after the Sun's gravitation stops.

Let $x(t)$ and $y(t)$ be a curve which describes the motion of an object in terms of time. Given a moment t_0 , we find the equation of the tangent to that curve at the point $(x(t_0), y(t_0))$.

We get a value t_1 such as $\Delta t = t_1 - t_0 \rightarrow 0$. Then, the tangent line at moment t_0 is described by the equation:

$$\begin{vmatrix} x & y & 1 \\ x(t_1) & y(t_1) & 1 \\ x(t_0) & y(t_0) & 1 \end{vmatrix} = 0$$

After expanding the determinant, and then dividing by $\Delta t \rightarrow 0$, we have:

$$x y'(t_0) - y x'(t_0) + y(t_0)x'(t_0) - x(t_0)y'(t_0) = 0$$

To test the formula above, we considered $x(t) = -c + a \cos(kt)$ and $y(t) = b \sin(kt)$, the equation of an ellipse. We obtain, by substitution:

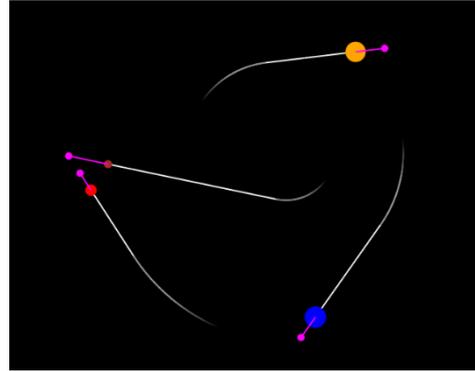
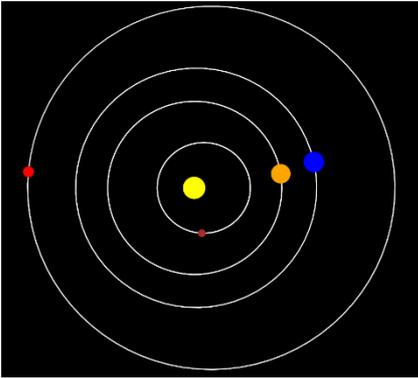
$$\frac{x \cos(k \cdot t_0)}{a} + \frac{y \sin(k \cdot t_0)}{b} - 1 + \frac{c}{a} \cos(k \cdot t_0) = 0$$

From where we easily find

$$\frac{x(x(k \cdot t_0) + c)}{a^2} + \frac{y y(k \cdot t_0)}{b^2} - 1 + \frac{c}{a} \cos(k \cdot t_0) = 0 \quad (3)$$

the equation of tangent for the ellipse centered at $(-c, 0)$ with an angular speed of $k \cdot t$.

So, according to our hypothesis, the removal of the Sun would cause planets to follow the line described by the equation above, process which is explained by the First Law of Dynamics, until they are involved in collision. We have used the formulas from (1.a) – (1.c), (2.a), (2.b) and (3) to create a simulator that shows the planets following their velocity direction as effect of the Sun removal.



The main idea in this simulator is to compute the positions of all the planets on the basis of their ellipse equations. At the removal of the Sun, each planet changes its trajectory by following the straight line with the direction of its velocity vector (tangent to its ellipse).

The simulation program offers just a theoretical model of planets motion, because of the hypothesis and the numeric approximations we made. The program uses the Aphelion and Perihelion constants to compute the parametric equation of each planet as described in (1.a) – (1.c). We observe that the program doesn't know that all the trajectories are based on the same equation structure like (2.a), (2.b). It treats them as equations of *any* continuous curves, so it uses the formula (3) to compute the tangent line equations and also to draw the tangent velocity vector [4]. For more accuracy, we set the parameter k of each planet to $f \cdot \frac{2\pi}{T}$, where T is the orbital period of the celestial body, and $f \in \mathbb{R}_+^*$ is a common arbitrary constant. This allows us to “speed up” the simulation by setting how much “steps” must a body perform per iteration (the value pointed by f).

Let's see how we can assemble the information presented to study the motion of a body B and, additionally, to look for interactions between bodies. We know the $\mathcal{A}, \mathcal{P}, T$ and k values for each planet. So, $\alpha(t) = k \cdot t = 2\pi f \frac{t}{T}$. Supposing the Sun is removed at moment t_0 , the body follows a trajectory line b described in the equation (3). Writing these equations for each $B_i \in \{Mercury, Venus, \dots\}, i = \overline{1,8}$, and solving the intersection of each two lines b_i, b_j , we can find if bodies B_i, B_j get into collision. [5]

We have created this simulation for the only purpose of having an idea of the phenomenon

starting from a very simple model, even if it partially alters the real situation. However, the information and the experience acquired until now was a crucial foundation of the next step in our research.

Rotation around the Sun (physics-based model)

Now we are going to predict the planets motion on the basis of the Laws of Mechanics.

Newton's Universal Attraction Law: Each two objects of masses m_1, m_2 attract each other with a force which is proportional to their masses and to the invert square of the distance between them:

$$F = G \frac{m_1 m_2}{r^2},$$

Where $G = 6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$ is the gravitational constant.

The vector form, where \vec{r}_1, \vec{r}_2 are the position vectors of each object:

$$\vec{F} = G \frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2).$$

Between any two objects, there are two forces of the same magnitude and opposite direction.

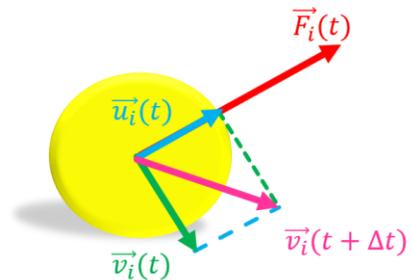
In a system of n bodies, of masses m_1, m_2, \dots, m_n , there will be a force $\vec{F}_{ij} = G \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|^3} (\vec{r}_i - \vec{r}_j)$ for each different $i, j \in \{1, 2, \dots, n\}$. Assuming $\vec{F}_i = 0$, we find that each body in the system is driven by the force

$$\vec{F}_i = \sum_{j=1}^n \vec{F}_{ij}, \forall i \in \{1, 2, \dots, n\}.$$

We will assign the vectors to a moment in time and use the second law of Newton:

$$\vec{F}_i(t) = m_i \vec{a}_i(t), \text{ from which } \vec{a}_i(t) = \frac{\vec{F}_i(t)}{m_i}. \text{ But } \vec{a}_i(t) = \frac{\vec{u}_i(t)}{\Delta t},$$

$$\text{then we find } \vec{u}_i(t) = \vec{a}_i(t) \Delta t = \frac{\vec{F}_i(t)}{m_i} \Delta t.$$



By Δt we refer to a small amount of time, so we can see that the velocity at moment $t + \Delta t$ is the sum of the velocity at moment t and the velocity corresponding to the momentum of the force at moment t , as shown in the picture.

Considering $\vec{v}_i(t_0)$ [6] the initial velocity of an object, we get the recurrence

$$\vec{v}_i(t + \Delta t) = \vec{u}_i(t) + \vec{v}_i(t) = \frac{\vec{F}_i(t)}{m_i} \Delta t + \vec{v}_i(t)$$

Now we have the system:

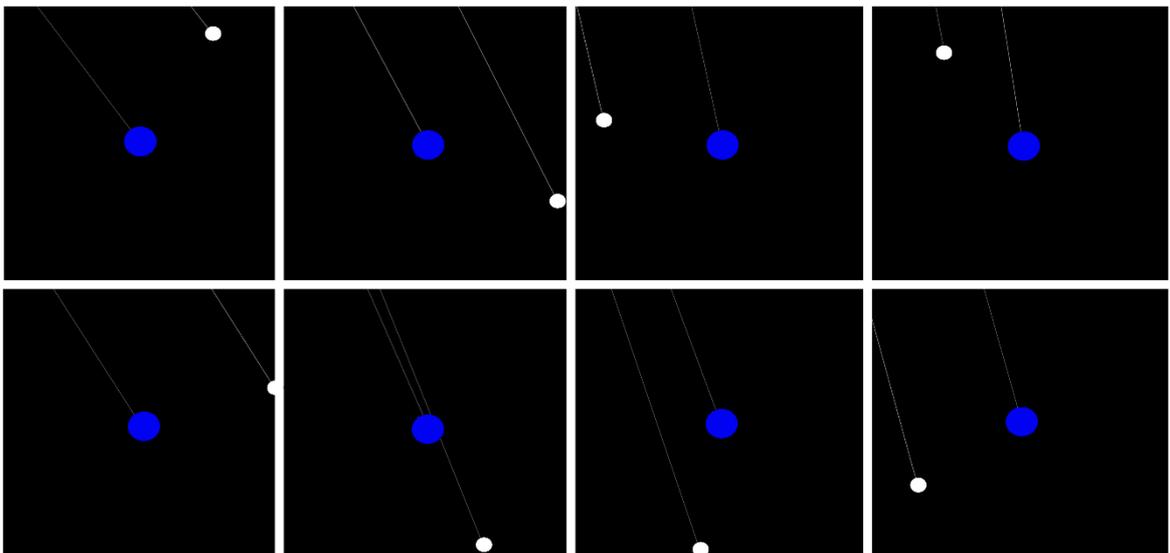
$$\left\{ \begin{array}{l} \vec{r}_i(t_0), \vec{v}_i(t_0), \forall i \in \{1, 2, \dots, n\} \quad (1) \\ \vec{r}_i(t + \Delta t) = \vec{r}_i(t) + \vec{v}_i(t)\Delta t, \forall i \in 1, 2, \dots, n \quad (2) \\ \vec{F}_i(t) = \sum_{j=1}^n G \frac{m_i m_j}{|\vec{r}_i(t) - \vec{r}_j(t)|^3} (\vec{r}_i(t) - \vec{r}_j(t)), \forall i \in 1, 2, \dots, n \quad (3) \\ \vec{v}_i(t + \Delta t) = \frac{\vec{F}_i(t)}{m_i} \Delta t + \vec{v}_i(t), \forall i \in 1, 2, \dots, n \quad (4) \end{array} \right.$$

Due to the difficulty to mathematically solve the system, we decided to create another simulator, this time trying as much as possible to illustrate the reality. The main pseudocode is:

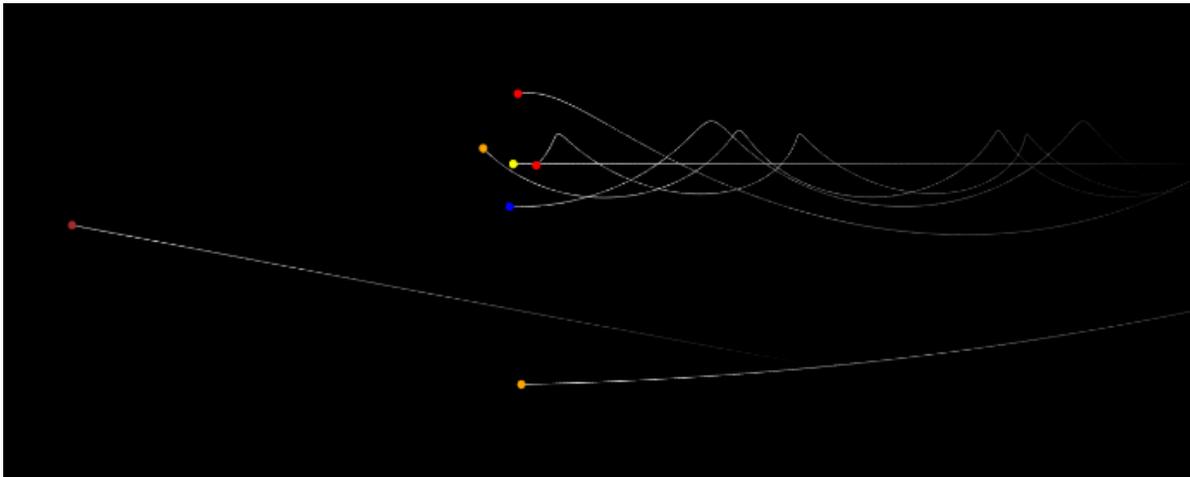
- › set global time to t_0
- › set Δt to a value near 0 (like 10^{-6}) (the smaller value of $\Delta t \rightarrow$ the smaller computational errors)
- › load the objects' given coordinates at t_0
- › **LOOP** (∞)
 - › For each planet:
 - › compute $\vec{r}_i(t + \Delta t)$ using (2) \rightarrow the next position of the object
 - › update the forces using the Newton's law of universal gravitation (3)
 - › compute $\vec{v}_i(t + \Delta t)$ using (4) \rightarrow the next value of velocity
 - › Increase global time with Δt
- › **END_LOOP**

The program can simulate any planetary system given the initial position and velocity of each object, as well as their masses and densities. The first three parameters are processed as described above, while the last one is used the development of graphics. The user has the option to move trough the whole system, zoom, pause, change simulation speed, select the objects and make them appear/disappear. Also, we can change the system as we want to discover other new facts, that are not in our research subject.

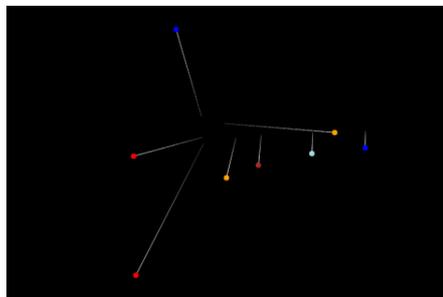
For example, we can add the Moon and check out how its trajectory changes around the Earth.



Also, we can see what happens of the planets if the whole solar system is moving in one certain direction, probably attracted by a bigger celestial body.

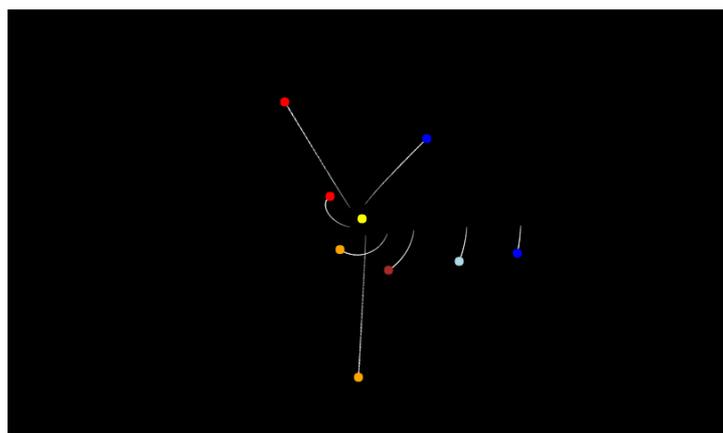


Now, returning to our subject, the simulation confirmed our hypothesis that, after removing the sun, the planets tend to keep their trajectories given by their momentum.



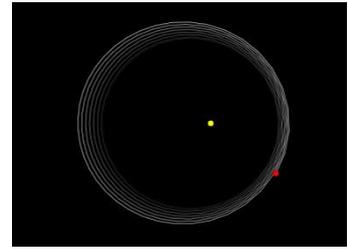
On the other hand, we were curious to find out what happens if the Sun suddenly reappears.

While some planets indefinitely leave the Solar System, the others are reshaping their orbits according to the law of gravity. There is no doubt that the way planets regain their elliptical trajectory is completely unexpectable, and depends on the position, velocity and mass of each planet at the moment of removing and putting back the Sun.



Conclusion

The subject is certainly a difficult one, due to its variety of cases which are almost impossible to study individually. We face a system in which the motion of each object affects the motion of all others, so it is hard to find a mathematical formula to explain this fact. Even the simplest system of one star and one orbiting planet doesn't follow an exact elliptical model trajectory, as it seems it is unlikely for the planet to reach a certain position in space twice. [7] However, we can „run” the system one step at a time to see its immediate evolution. That's the reason why we created a program to check whether our expectations were correct. As we can see, the idea of following the direction of orbital velocity was confirmed both by geometrical and physical model, so this is the most appropriate answer to our subject.



All we can do is just to admire the power of the nature to find its own balance. [8]

EDITION NOTES

[1] It is not necessary that the angular speed is constant. It is enough that the angle $\alpha(t)$ is a known function.

[2] In this equality and below, the letters \mathcal{A} and \mathcal{P} denote numbers, while, above, they denote points. Everything is certainly understandable, but different styles for these letters would be preferable.

[3] The First Law of Dynamics should be mentioned here.

[4] The formula (3) provides only the line of the velocity vector. In order to have also its direction and magnitude, we must also consider the values in t_0 of the derivatives $x'(t)$ and $y'(t)$.

[5] The two lines might intersect, but still there could be no collision. In order that the two planets collide, not only their lines must intersect, but, in the motion, the two planets must reach the intersection point at the same moment.

[6] Also here, it is not clear how $\vec{v}_i(t_0)$ is determined. See Note [4].

[7] It should be said why “it is unlikely for the planet to reach a certain position in space twice”. According to the Kepler’s First Law, it should.

[8] Many philosophers would hold that this sentence is a rather controversial one. Just a first question could be: is the balance reached in “Nature”? or in our description of it?