A partition without end or without hunger?

Gianolla Matteo (3D), Giuliani Marco (2H), Mion Pietro (2H),
Scasso Tommaso (4G), Sinatora Pietro (4G), Zuliani Alessandro (2H)

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School: Liceo Scientifico Giordano Bruno (Venezia-Mestre)
Teacher: Lucia Rossi
Researchers: Francesco Rossi, Alberto Zanardo (Università degli Studi di Padova)

1 The subject

In Autumn, a population of squirrels stocks up on hazelnuts to pass the Winter. Each squirrel collects his personal stock of hazelnuts. To make sure every squirrel has the same amount of hazelnuts, they made up a partition system: when two squirrels meet, they compare their stocks. The squirrel who has less hazelnuts receives by the other the same number of nuts he owns. This method goes on until they own the same amount of hazelnuts.

Questions:

- Are there any situations where this partition never ends?
- If this partition ends, how many steps does it take to get to end?

2 Notations

We call the two numbers of hazelnuts as $(a;\ b)$ with $a, b \in \mathbb{N}$, a is the amount of nuts of the first squirrel and b is the amount of the second one.

To be clearer we are going to make two examples:

Example 2.1. In this case the partition does not end:

We start with	$(7;3)$
The first squirrel gives the other 3 hazelnuts, and we obtain	$(4;6)$
After that the first one receives by the other 4 nuts, we obtain	$(8;2)$
If we go on with this method, we will arrive back at	$(6;4)$
and the partition will not end.	

Example 2.2. In this case the partition ends:

$$(31;1)$$
$$(30;2)$$
$$(28;4)$$
$$(24;8)$$
$$(16;16)$$

First of all we need to establish some useful definitions.

Definition 2.1. We have a step when a squirrel gives the other some nuts: a couple $(a;\ b)$, after a step, becomes $(2a;\ b-a)$, if $b > a$ or $(a-b;\ 2b)$ if $a > b$.

Definition 2.2. A couple $(a;\ b)$, with $a, b \in \mathbb{N}$, is solvable if, after a finite number of steps, the squirrels will have the same number of hazelnuts.

3 Solution

The aim of the problem is to find the cases which are solvable and define the number of steps necessary to solve them.

3.1 Trivial cases

While we were working on to find a solution, we found out some trivial cases that we summarize below:

- if $a+b$ is odd the couple is not solvable
- if $(a;\ b)$ is solvable, then $(b;\ a)$ is solvable
- if $a = 0$ and $b \neq 0$ the couple is not solvable
- if $a = b$ the couple is trivially solvable

From now on we will consider only the non-trivial cases.

3.2 Solvable couples

First of all, we demonstrate an useful lemma.

Lemma 3.1. *In each step after the first both number are even.*

Proof: We can assume without loss of generality that $b > a$, so the first step will become:

$$(a;\ b)$$
$$(2a;\ b-a)$$

$a+b$ is even, so a and b have to be both even or both odd; so $b-a$ is even and $2a$ is even. □

Now we observe the relationship between some couples.

Theorem 1. *For all $k \in \mathbb{N} \setminus \{0\}$, the couple $(a;\ b)$ is solvable if and only if $(ka;\ kb)$ is solvable.*

Proof: There's a one-to-one correspondence between the steps of the case $(a;\ b)$ and the steps of the case $(ka;\ kb)$:

Case $(ka;\ kb)$	**Case** $(a;\ b)$
$(ka;\ kb)$	$(a;\ b)$
$(ka-kb;\ 2kb) = \big(k(a-b);\ k(2b)\big)$	$(a-b;\ 2b)$
...	...

So if we can solve a couple, we will be able to do that also for the other one. □

Corollary 1.1. *If k divides both member of the couple $(a;\ b)$ in one step, k will divides both of them for all the next steps.*

We have noticed that when the sum of the couple is equal to a power of two, the couple is always solvable. We have demonstrated it by induction in the next theorem.

Theorem 2. *If* $a+b=2^n$*,* $n \in \mathbb{N}$*,* $n \geq 2$*,* $(a;\ b)$ *can be solved in at most* $n-1$ *steps.*

Proof: By induction on n. a) The statement is true for $n=2$: the possible cases are $(3;\ 1)$ or $(2;\ 2)$

- $(2;\ 2)$ is a trivial case
- $(3;\ 1)$ finishes in one step: $(3;\ 1)-(2;\ 2)$.

b) We prove that if the statement is true for $n-1$, it will be true for n:

We can assume without loss of generality that $b > a$.

If $a+b=2^n$ then $b=2^n-a$. Then$(a;\ 2^n-a)$ becomes $(2a;\ 2^n-2a)=\big(2a;\ 2(2^{n-1}-a)\big)$.

Since the couple $(a;\ 2^{n-1}-a)$ is solvable by the induction hypothesis and according to Theorem 1 (with $k=2$) the couple $(2a;\ 2^n-2a)$ is solvable at most $n-2$ steps.

So the maximum number of steps is at most $n-1$. □

Next, we are going to extend the solvability to other couples and demonstrate that there are no more solvable couples.

Theorem 3. *The couple* $(a;\ b)$ *can be solved if and only if it exists an odd* k *such that* $a+b=2^n k$ *and* k *divides* a *and* b *(*$k \mid a, b$*).*

Proof: The first part of the theorem is trivial: according to Theorem 1 and Theorem 2, if $a+b=2^n k$ and $k \mid a, b$, the couple $(a;\ b)$ is solvable.

Now we demonstrate that if $(a;\ b)$ is solvable then $a+b=2^n k$ and $k \mid a, b$.

We are going to divide the proof in three parts.

a) We prove that if (a, b) is solvable (excluding trivial cases), then $a+b=2^2 h$, with $h \in \mathbb{N} \setminus \{0\}$. If the couple $(a;\ b)$ is solvable, we will get, in the last step, a situation like $(z;\ z)$. Without considering trivial cases, compared to the previous step, one of the amount of hazelnuts has doubled, so it exists $h \in \mathbb{N} \setminus \{0\}$ such that $z=2h$; so $a+b=2z=4h$ and $a+b=2^2 h$.

b) Now we prove that if n is the maximum exponent of 2 such that $a+b=2^n k$, then k divides a and b in all the previous steps.

If $(a;\ b)$ is solvable and $a+b=2^n k$, the last step is $(2^{n-1}k;\ 2^{n-1}k)$, where both numbers are divisible by k.

The previous step has to be $(2^{n-2}k;\ 2^{n-2}k+2^{n-1}k)=(2^{n-2}k;\ 3\cdot 2^{n-2}k)$, which is still divisible by k. We can observe that this configuration is such as $(\alpha 2^m k;\ \beta 2^m k)$ where α, β are odd.

We want to demonstrate that the previous step is such as $(\alpha' 2^{m-1}k;\ \beta' 2^{m-1}k)$ where α', β' are odd and where both number are still divisible by k.

We can suppose without loss of generality that $\alpha > \beta$. The couple $(\alpha 2^m k;\ \beta 2^m k)$ comes from an only one configuration: $((2\alpha+\beta)2^{m-1}k;\ \beta 2^{m-1}k)$(1); we notice that the coefficients $2\alpha+\beta$ and β are still odd.

c) According to Lemma 3.1 this process can continue as long as $m \geq 1$; so $k \mid a, b$ in all the previous steps (2). □

3.3 Calculating the steps

The next aim is to demonstrate how many steps a couple requires to be solved.

We divided our reasoning in two theorems, a simple one and a general one.

In the simple case, both members of the starting couple are odd.

Theorem 4. *The number of steps to solve a solvable couple* $(\alpha k;\ \beta k)$*, with* α, β *coprime and where* k *is odd, is* $\log_2(\alpha+\beta)-1$.

Proof: Considering Theorem 1, the couple $(\alpha k;\ \beta k)$ has the same steps as the couple $(\alpha;\ \beta)$.

To make the couple solvable, by Theorem 3, $\alpha+\beta=2^n$.

Considering Lemma 3.1, after the first step both numbers are divisible by 2^1.

After another step, both numbers are divisible by 2^2.

Generally, after $n-1$ steps, both numbers are divisible by 2^{n-1} (3). The only two positive numbers which are divisible by 2^{n-1} and which have sum 2^n, are 2^{n-1} and 2^{n-1}. The two numbers are equal, so the couple got solved with $n-1$ steps, that can be written as $\log_2(\alpha+\beta)-1$. □

We conclude our reasoning with the general case.

Theorem 5. *The number of steps to solve a solvable couple* $(\alpha 2^m k;\ \beta 2^m k)$, *where* α, β *are coprime and where* k *is odd, is* $\log_2(\alpha+\beta)-1$.

Proof: Considering theorem 1, the couple $(\alpha 2^m k;\ \beta 2^m k)$ has as many steps as the couple $(\alpha;\ \beta)$.

If $\alpha 2^m + \beta 2^m = 2^n$, then $\alpha+\beta=2^{n-m}$.

According to the previous demonstration, the steps are $n-m-1=\log_2(\alpha+\beta)-1$. □

4 Conclusion

The couples which are solvable are such as $(a;\ b)$ with $a, b \in \mathbb{N}\setminus\{0\}$, $a+b=2^n k$ and $k \mid a, b$. This couples are solvable in $n-(m+1)$ steps, where m is the maximum exponent of 2 such that $2^m \mid a, b$.

Editing Notes

(1) In fact $(\alpha 2^m k;\ \beta 2^m k)$ may also come from $(\alpha 2^{m-1} k;\ \big((\alpha+2\beta)2^{m-1}k\big)$ at the previous step (and the order between α and β does not matter here). But the conclusion that the coefficients of $2^{m-1}k$ are odd still holds in this case.

(2) When $m=1$, we get a couple $(\alpha k;\ \beta k)$ of odd integers; indeed, k is odd since $a+b=2^n k$ where n is the greatest integer such that 2^n divides $a+b$. Then, according to Lemma 3.1, there is no possible previous step. It follows that either this couple is $(a;\ b)$, or $(a;\ b)$ appeared earlier in the the sequence of couples $(\alpha 2^m k;\ \beta 2^m k)$ and thus in all cases $k \mid a, b$.

(3) It must be shown that the process *does not stop* before step $n-1$: indeed, we start with α and β which are odd since they are coprime and $\alpha+\beta=2^n$, and it follows from the proof of Theorem 3 (b) that $n-1$ steps are required to get $(2^{n-1};\ 2^{n-1})$ from a couple of odd numbers.