Musical improvisations

Year 2023 – 2024

MESNIER Domitille, Year 11

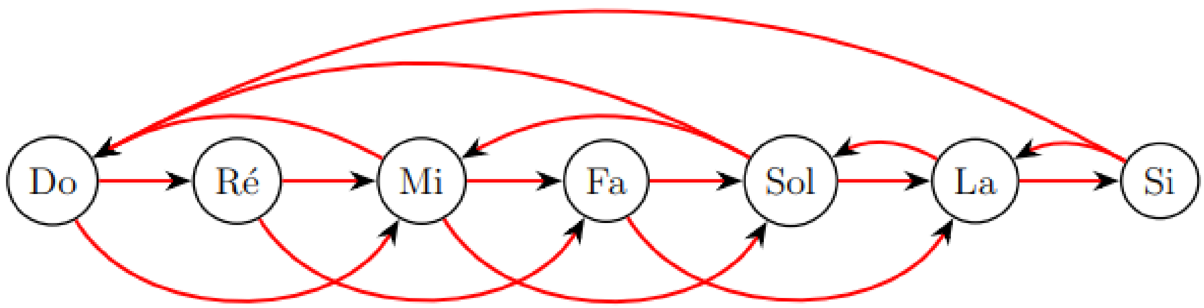
Supervised by: CASTAGNEDOLI Sebastien, FORT Elodie and PASTEUR Laurent.

School: Institute of Emmanuel d'Alzon, at Nîmes.

Researcher: DUMONT Serge, Lab: laboratory of mathematics and physics (LAMPS) University of Perpignan Via Domitia (UPVD).

1. Topic presentation

This topic is about musical improvasition. We must determine how many different tunes are possible with notes and combinations giving. Knowing that we will start this improvisation with a C and that there isn't twice the same combinations with two notes in every tune so that's there isn't twice the same arrows (if we write C and D in this order, we can't get C and D twice but we can write D and C in this order). The drawing below that we can call a graph (a diagram with points called vertices: C, D, E, F, G, A and B, linked or not by segments called edges →) enable us to do know what are the possible combinations. Indeed, the arrows show us the correct combinations with the notes.



2. Presentation of the conjectures and the final results

First, we will discover the rough method. Then, we will make a more precise study thanks to a case-by-case analysis.

It's difficult to find a precise result because the final number is big. But, we can use different methods.

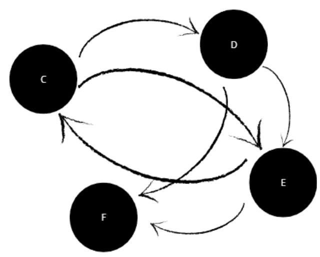
Firstly, we know that the maximum number of the notes is 7 and we write N any note. Knowing that each notes are repeated an unknown number but the same combinations with two notes aren't repeated twice (the note C can be repeated an unknown number but the combination C and D can't be repeated twice). So, we have: (N_1; N_2), a combination.
For N_1, we have seven possibilities: C, D, E, F, G, A and B. Therefore, for N_2, we have $7-N_1=6$ possibilities. We have seven and six possibilities because the tunes don't begin necessarily with the notes C. They can begin by seven different notes.
When we multiply 7 by 6, we find 42 which is the total of the possible numbers of tunes with two notes (N_1 and N_2) amoung the seven giving. As there are two notes in each tunes and that the total of tunes is forty-two, we have forty-two times tunes with two notes. So, 2^{42} is the number of graphs with seven notes which aren't repeated twice.

The second method is the same but we use arrows. We know that the maximum number of arrows is 16 and we write F any arrow. Knowing that each arrows aren't repeated twice (the arrow which links C and D can't be use two times). So, we have: (F_1; F_2), a combination.
For F_1, there are sixteen possibilities and for F_2, there are $16-F_1=15$ possibilities. We have sixteen and fiveteen possibilities because the tunes don't begin necessarily with the arrows which start at C.
When we multiply 16 by 15, we find 240 which is the total of the possible numbers of tunes with two arrows so that's three notes (F1 and F2) amoung the sixteen giving. As there are two arrows in each tunes and that the total of tunes is two hundred forty, we have two hundred forty times tunes with two arrows so that's three notes. So, 2^{240} is the number of graphs with sixteen arrows which aren't repeated twice.

3. Article text

To find the possible results of this topic, we can use different methods such as the graph theory and the matrix.
First, we can simplify the giving graph above. For example, if we keep the notes C, D, E and F (the first four notes) we must keep the same arrows which link the different notes.

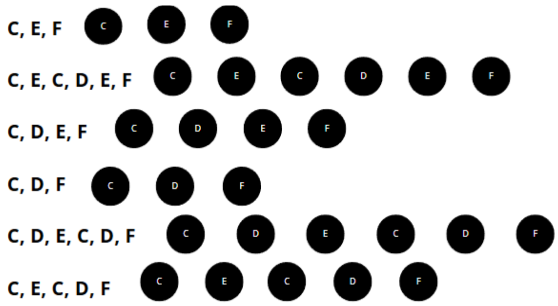
We know that a graph, known as G, has different points called vertices, written V, and different edges, written A, which relate vertices. In the graph, the vertices are the notes(C,D,E and F) and the edges are the relationships between the notes (the arrows). As the edges represent the directions, so it's an oriented graph.



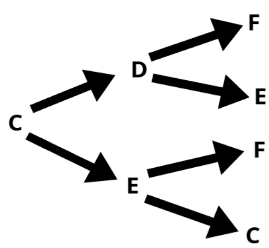
So, here are some important definitions to understand this topic.
- This graph is oriented because the edges are arrows which show a direction, so an edge is called an arch written A.
- An arch is defined by an organized pair (Av_1; Av_2) of vertices because the arch start of a vertice (Av_1) and reach of another vertice (Av_2), so the arch links two vetices.
- A length of a chain is equal to the number of the edges which is made up of it. Indeed, the more edges the chain has, the taller the chain is and vice-versa.
- Let's consider G=(V,A) an oriented graph: a path of a vertice v_1 toward a vertice v_k which is a row (v_1; v_2...v_k)

of the vertices like two consecutive vertices v_i; v_i+1 linked by an arch of G. For example, the path C, D, E and F is a row of vertices because these are vertices followed one after the other. C and D are two consecutive vertices which are linked by an arch of the graph G.
- v_1 is the origin and v_k is the end of the path. In our exemple, C is a vertice (v_1) which is the origin of the path and F is a vertice (v_k) which is the end of the path.
- The length of the path is the number of the arch in the path, that is (v_1; v_2...v_k)-1 because there is always one vertice more that the number of the arch. Indeed, an arch links two vertices...
- A path may be made of only one vertice and be of length 0 because there aren't arches which link only one vertice. If there is one vertice, there aren't arch so the length is 0.
- A track is a path of length that is not equal to 0, because there is least one arch, whose the origin and the end which are the same because it's closed.
- A loop is a tour of length 1 because it's closed and there is one arch.
- A path is elementary if it doesn't go twice by the same vertice (except for the origin and the end for a tour).
- Let's consider G a graph; if x and y are two vertices of G, the distance (written d) of x at y marked d(x; y), is the length of a shorter chain of G link x at y. Indeed, we must always follow the shorter path between two vertices which we would like link.

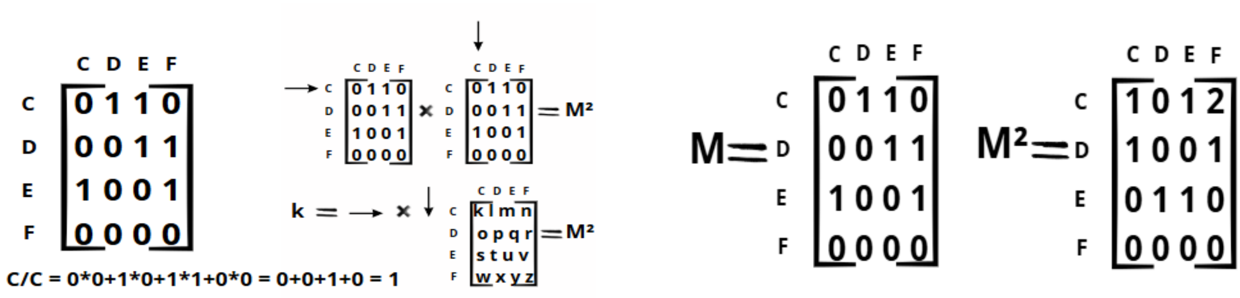
The first possibility is to look for the different possible tunes with the notes. We can write the row of the notes if they are a link with the previous note. But with these tunes there isn't any relationship between two notes which are repeated twice. We can begin by the shorter tunes or by the bigger tunes until every combinations are used.
This method is long and the possible errors are important.



Then, the tree is another way to solve the problem. The tree permit to calcul probabilities; to make this, we start by an origin called root of the tree and with the arrows called branches which link the notes called leafs and nodes: every possible events. Indeed, the first note which is C, is the root of the tree. We relate with arrows the other notes which are related to C. We count the possible notes to follow the arrows and their direction. For example, C is linked with two notes: D and E, and D is linked with two notes: F and E. So, here, we have a tune with three notes C, D and F.
To make a tree, we consider C, the origin, after we relate it to D and to E separately which have a relationship with C. We continue to follow the same method which links the notes without the same notes being twice in the same order (if we write C and D in this order, we can't get C and D twice but we can write D and C in this order).



The last method is with the matrices. The matrix with the size (n;p) is a “table” of the size *nxp* where we define a value: an element of A in every cases. n is the number of the lines and p is the number of the columns. Every cases of the matrix must be filled. To know, the total of every coefficient of the adjacency matrix of an oriented graph is equal to the number of the arches of this graph (the coeficient dc;f is equal to the distance between the vertices c and f).
Let's consider G a graph and M it's adjacency matrix: the number of the chains of length n linking the vertice c at the vertice f given by the term of subscript c, f of the matrix M^n.
To make a matrix, we place the name of the vertices of the graph horizontally and vertically in the same order, outside the matrix (see below). After, we place a 0 when there isn't an arrow which links the note of the graph horizontally and the note vertically in the matrix. And we place a 1 when there is an arrow which links the note in the graph horizontally and the note vertically in the matrix.
To find the matrix to the power two with the matrix 1, we multiply the line C with the column C and we find the number which is in the square C (horizontal) and C (vertical). To find the other number for every square, we continue like this; we multiply the line C with the column D and we find the number which is in the square C (horizontal) and D (vertical).



Let's watch the graph and the matrix. When we add the numbers from the column C, in the matrix, we can find the same number as the one of the arrows reaching the vertice, in the graph (in the column of the matrix, the number of C is equal to one and in the graph, there is one arrow which reach at C). And when we add the numbers from line C, in the matrix, we can find the same number as the one of the arrows starting at the vetice, in the graph (in the line of the matrix, the number of C is equal to two and in the graph, there are two arrows which arrive at C).
So, the column in the matrix is equivalent to the arrows which reach the vertice affected in the graph. The line in the matrix is equivalent to the arrows which start at the vertice affected in the graph.

column c = 0 + 0 + 1 + 0 = 1
line c = 0 + 1 + 1 + 0 = 2

Finally, we know that the sum of the coefficient of the matrices M^1, $M^{2,}$ M^3 and 7 is the number of the tunes which length is equal to four or less; the maximum number of notes is four. Indeed, we have keeped only four notes.

4. Conclusion

So, in this topic, the final result is a big number, but, we can use two methods which can make the research easier. We can find the beginning of a result with the matrix seven by seven, we find 40 520 tunes of eleven notes amoung the seven giving (see below).

	C	D	E	F	G	A	B
C	1036	436	1056	625	1068	852	360
D	1014	439	1042	615	1066	833	363
E	1470	597	1498	877	1467	1220	489
F	1001	417	1021	604	1023	824	344
G	1417	617	1457	866	1504	1157	516
A	971	384	984	572	945	814	308
B	820	354	847	502	866	668	298