LIGHT TRAP

School year: 2018-2019
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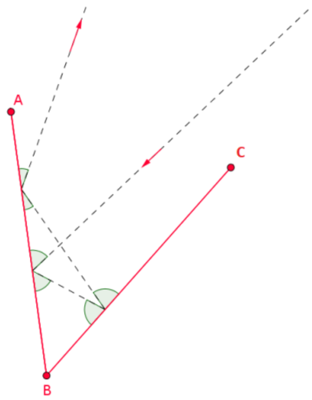


INTRODUCTION

Throughout the school year 2018/2019 we were taking part in a project Erasmus+ called "Maths & Languages". Our group worked on the subject called "Light Trap".

OUR TASK

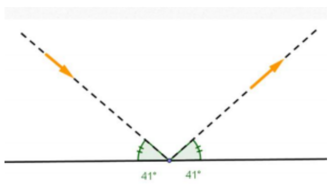
In the picture you can see a ray of light that slides into a figure f and then emerges after 3 reflections.

Task 1: Determine points A, B, C and a ray of light so that one could obtain the maximum of reflections. Is it possible that the ray never comes out?

Task 2: To do the research with some other potential light trap.

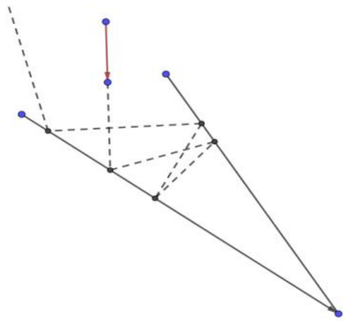
How is a reflection made?

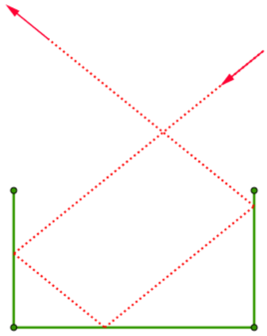
The angle of incidence equals the angle of reflection.

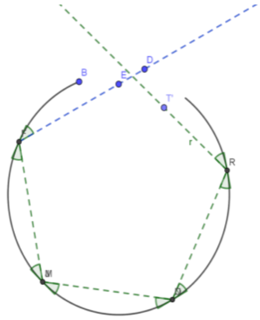


EXAMPLES

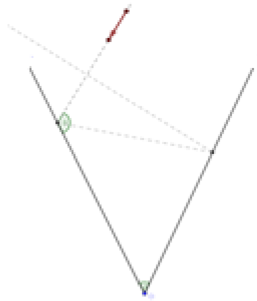
We decided to start with three kinds of figures: angle, well and arch. Below you can see some examples.







UNBOUNDED ANGLE TRAP



Let's start with a figure f composed of two half-lines coming out from one point - vertex O, intersecting at the angle α.
By β we denoted the angle between the ray and one side of the angle. α is an angle between two half lines.

Does the number of reflections depend on β/α or on β-α ?

In Geogebra we were trying to find dependence between β and α. After that we set up the table with our observations:

β	130	130	130	130	130
α	50	60	40	30	10
β-α	80	70	90	100	120
β:α	2.6	2.2	3.3	4.3	13.0
n	3	3	4	5	13

Conjecture: The number of reflexions depends on β/α.

Theorem 1:

Let $0 < \alpha < 180°$, $0 < \beta < 180°$. The number of reflections is finite and equal to $\left\lceil \frac{\beta}{\alpha} \right\rceil$. [1]

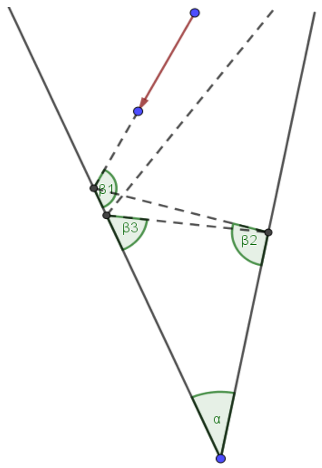
Examples for ceiling function:

$\left\lceil \frac{120}{40} \right\rceil = 3 \qquad \frac{120}{40} = 3$

$\left\lceil \frac{120}{41} \right\rceil = 3 \qquad \frac{120}{41} = 2,92$

Proof

By β2, β3 … we determined the angles related to the falling ray of light for subsequent reflections.



Lemma 1:

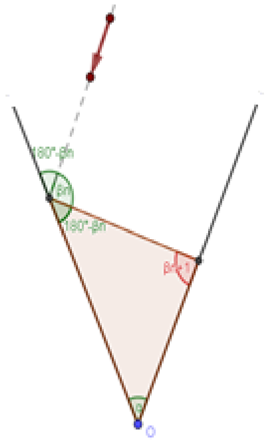
There will be the next n+1–th. reflection if and only if $\beta_n > \alpha$.

If $\beta_n > \alpha$, then $\beta_{n+1} = \beta_n - \alpha$.

Proof:

$(180 - \beta_n) + \alpha < 180^o \Leftrightarrow \beta_n > \alpha$

If $\beta_n > \alpha$, then $\beta_{n+1} = 180^o - [(180^o - \beta_n) + \alpha] = \beta_n - \alpha$

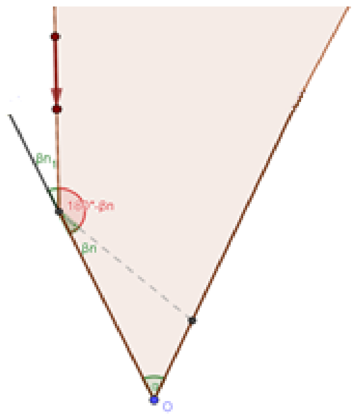


Lemma 2 :

There will be no next n+1-th reflection (the ray «is coming out»)
if and only if $\beta_n \leq \alpha$. [2]

Proof:

$(180^o - \beta_n) + \alpha \geq 180^o \Leftrightarrow \beta_n \leq \alpha$



Proof of the theorem:

We need to prove that the number of reflections is finite and equal to $\left\lceil\frac{\beta}{\alpha}\right\rceil$.

(1) If there are at least n reflections, we have:

$\beta_1 = \beta > 0$

$\beta_2 = \beta_1 - \alpha = \beta - \alpha > 0$

$\beta_3 = \beta_2 - \alpha = \beta - \alpha - \alpha = \beta - 2\alpha > 0$

……………………….

$\beta_n = \beta - (n-1)\alpha > 0$

If there are at least n reflections, then:

$\beta_n > 0$

$\beta - (n-1)\alpha > 0$

$\beta > (n-1)\alpha$

$\beta/\alpha > n-1$

$n < \beta/\alpha + 1$

So, the number of reflections **has to be finite.**

(2) Let n be the exact number of reflections. Thus there was no n+1-th reflection, so (by Lemma 2):

$\beta_n = \beta - (n-1)\alpha \leq \alpha$

$\beta \leq n\alpha$

$n \geq \beta/\alpha$

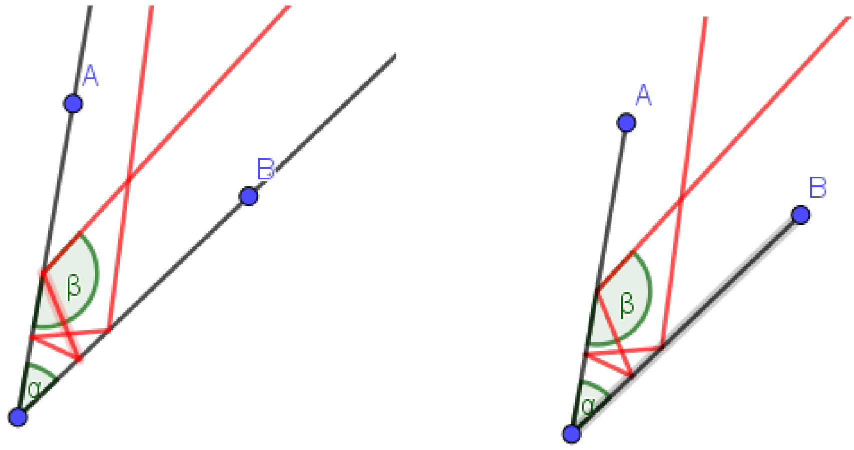
(3) By (1) and (2) we obtain: $\beta/\alpha \leq n < \beta/\alpha + 1$ so $n = \lceil\beta/\alpha\rceil$.

BOUNDED ANGLE TRAP

How to obtain any number of reflections?

Suppose that we want to obtain n reflections.

- Create an unbounded angle trap with n reflection . Task to the Readers: Please try to guess why α+β could be greater than 180°,
- Cut the sides "above" all the reflections,
- If the size of our trap is too big – use homothetic transformation of the center O.



INFINITE WELL

The figure we will be working on, is a well whose sides are half lines parallel to each other and perpendicular to the base. As the sides are infinitely long, the number of reflections is one* or infinite** (as if the ray does not fall in the corner).

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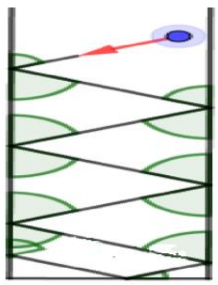

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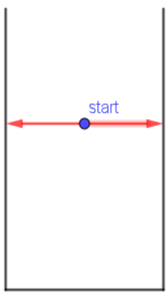

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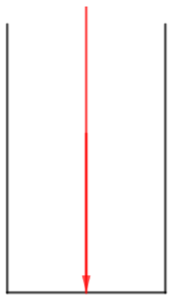
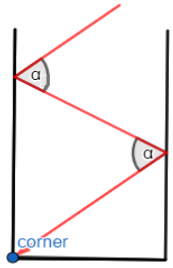


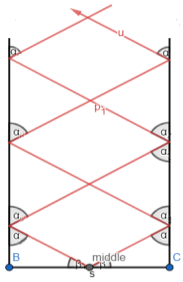
FINITE WELL

We have to find out whether the number of reflections is finite.

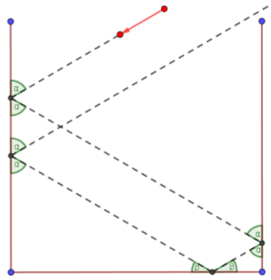
1)



2)



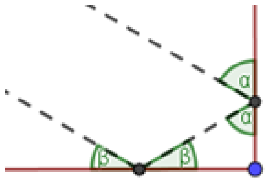
3)



There are three possible cases:

1. The ray falls into the corner. The light is “scattering”.
2. The ray falls in the middle of the base. The number of reflections at the left side is the same as at the right side. The marked angles are of the same size.
3. The ray does not fall in the middle of the base. The marked angles are also of the same size (see the next slide). Basing on the angle of the reflection, the size of the well and the place where the ray reflects off, we can count the number of reflections, but we haven’t done that.

Observing the last picture concerning the finite well, we have found out that the ray formed a parallelogram. Its field can be described by a formula: 360° = 2(180° -2α)+ 2(180°-2β). From this formula it follows that 90°= α+β.

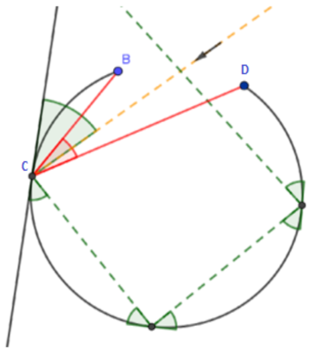


We know that the ray falling in the corner (which in our well is 90°) has a complement that is equal to α. We know that after the reflection α=>β and β=>α. It means that the ray will reflect in very precise way.

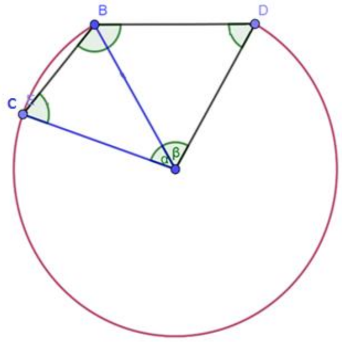
ARCH TRAP

Notation

ANGLE OF INCIDENCE
γ – angle between the tangent and incidence light ray



ANGLE α
α- the angle between ray OB and ray OC,
where C is the incidence point (O is the centre of the circle that contains our arch)



ANGLE OF THE WIDTH
β– angle width of the opening of the arch

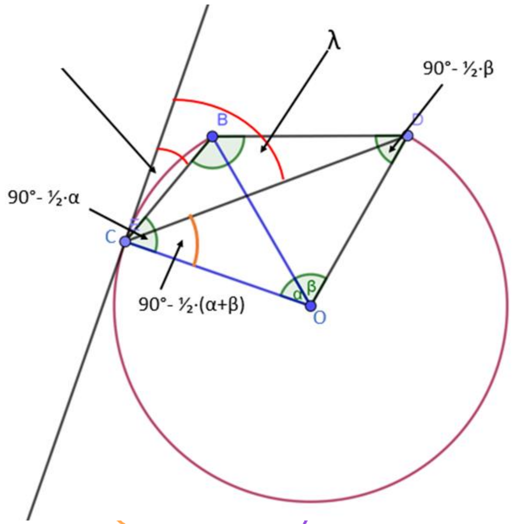
<u>Dependence between the angle of incidence and the angle of the width</u>

Our first task was to find the dependence between the angle of incidence and the angle of the width of the entrance to the arch.

θ

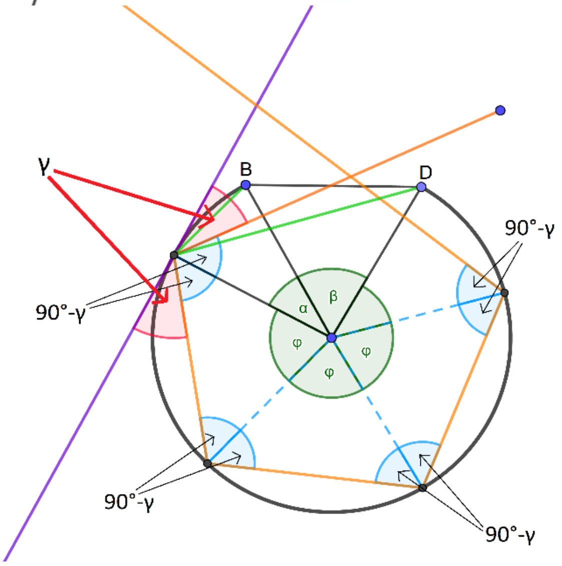
We had to find the minimum and the maximum angle of incidence, so that the light could enter the arch. The minimum angle is marked as θ and it's the difference between the right angle (the one between the radius CO and the tangent) and the one marked as BCO.

The angle $OCB = \frac{180°-\alpha}{2} = 90° - \frac{\alpha}{2}$,
so $\theta = 90° - \left(90° - \frac{\alpha}{2}\right) = \frac{\alpha}{2}$.
Similarly: $\lambda = \frac{\alpha+\beta}{2}$.
Finally $\frac{\alpha}{2} < \gamma < \frac{\alpha+\beta}{2}$.



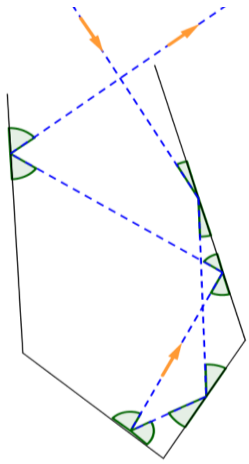
φ is the angle between two circle radius drawn to two consecutive points of incidence on the arch.
We can see that:
φ=180°-2(90°-γ) = 180°-180°+2γ = 2γ.

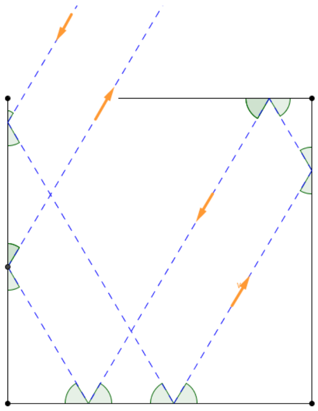
[3]
If we assume that the light ray will go out through the opening after one circulation then the number of reflections can be expressed by k+1,
where k is given by the formula k = $\left\lfloor \frac{360°-\alpha-\beta}{2\gamma} \right\rfloor$,
so the number of reflections will always be finite.



<u>Remark:</u>
Our assumption that there will be only one circulation is equivalent to the following condition:
360°-α> (k+1)2γ.

OTHER EXAMPLES



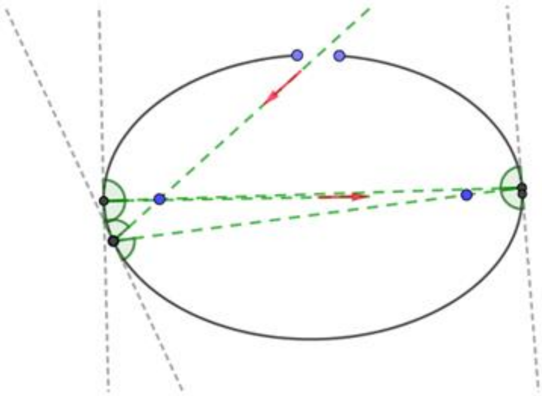


Open pentagon

Open square

ELLIPSE

Our researcher, Yohann Gemzer, advised us to try with ellipse. He was right, what we showed on the picture.



It holds due to the theorem that if a light source passes through one focus of elliptic mirror, all other light rays on the ellipse are reflected to the second focus.

OVERVIEW

We have studied the ray of light sliding into an angle, well, arc and ellipse. We encourage you to study other shapes.

EDITING NOTES

[1] High school students are in general not familiar with the symbols $\lceil .. \rceil$ and $\lfloor .. \rfloor$. They should have been defined.

[2] This lemma is just the first part of Lemma 1. If two conditions are equivalent, their negations are equivalent as well.

[3] This final part of the arch trap case deserves a more detailed and careful development.

- The case in which $\frac{360° - \alpha - \beta}{2\gamma}$ is an integer number – so that the light ray meets the point D – should have been discussed. Will the ray scatter when it meets D? Or there will be another reflexion? In this second case, the number of reflexions is no longer $k + 1$.
- The sentence "so the number of reflexions will always be finite" is not proved. It might be false. Perhaps, the sentence refers only to the situations that we are considering, in which the light ray is assumed to go out after one circulation. But in this case the claimed result is an obvious consequence of the assumption.
- The condition $360° - \alpha > (k + 1) \cdot 2\gamma$ should have been studied and commented more in detail, in order to determine more explicit conditions in terms of α, β and γ. To this aim, it may be useful to express k as $\frac{360° - \alpha - \beta}{2\gamma} + \varepsilon$, with $0 \leq \varepsilon < 1$.