Let’s change the rules !

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Presentation of the topic

Knowing if a number can be divided by 5 is easy: just make sure that the unit digit of its decimal representation is. The same thing if you want to divide it by 2 or by 10. If you want to divide it by 7 or by 13, things get less trivial.

Could we change the rules? For example, could we write numbers in order to check the divisibility by 7, it is enough to consider the last digit? Can we use several chosen rules simultaneously?

1. Divisibility by 7

Rule:

Any number is divisible by 7 if and only if the absolute difference between twice the unit digit and the number formed by the rest of the digits is divisible by 7.

Example:

The number 32928 is divisible by 7 because when we apply this criterion in a recursive manner we have:
32928 ⋮ 7 ⟷ (3292 – 2 · 8) ⋮ 7 ⟷ 3276 ⋮ 7 ⟷ (327 – 2 · 6) ⋮ 7 ⟷ 315 ⋮ 7 ⟷
(31 – 2 · 5) ⋮ 7 ⟷ 21 ⋮ 7 (true)

Proof:

We have:
$N = 10^n a_n + 10^{n-1} a_{n-1} + 10^{n-2} a_{n-2} + \cdots + 10^2 a_2 + 10 a_1 + a_0$
Let:
$N = 10^n a_n + 10^{n-1} a_{n-1} + 10^{n-2} a_{n-2} + \cdots + 10^2 a_2 + 10 a_1 + a_0 = 7k$

for some integer k. Then:

$$N = 10(10^{n-1} a_n + 10^{n-2} a_{n-1} + 10^{n-3} a_{n-2} \ldots + 10 a_2 + a_1) + a_0 = 7k$$

Add $20a_0$ and substract on the equation to get:

$$N = 10(10^{n-1}a_n + 10^{n-2}a_{n-1} + 10^{n-3}a_{n-2} \dots + 10a_2 + a_1) + 20a_0 - 20a_0 + a_0$$
$$= 10(10^{n-1}a_n + 10^{n-2}a_{n-1} + 10^{n-3}a_{n-2} \dots + 10a_2 + a_1 - 2a_0) + 21a_0 = 7k$$

$$7k - 21a_0 = 10(10^{n-1}a_n + 10^{n-2}a_{n-1} + 10^{n-3}a_{n-2} \dots + 10a_2 + a_1 - 2a_0) \equiv 0 \pmod 7$$

This is equivalent with:

$$10(\overline{a_n a_{n-1} a_{n-2} \dots a_2 a_1} - 2a_0) \equiv 0 \pmod 7$$

Therefore, since 10$\equiv 3 \pmod 7$, for N to be divisible by 7, it must be true that

$$\overline{a_n a_{n-1} a_{n-2} \dots a_2 a_1} - 2a_0 \equiv 0 \pmod 7$$ (1)

2. Divisibility by 13

Rule:
Any number whose sum of four times the unit digit and the number formed by the rest of the digits is divisible by 13 is itself also divisible by 13. (2)

Example:
The number 65 286 is divisible by 13 because when we apply this criteria in a recursive manner we have 13|6528+6 · 4 ↔13|6552 ↔ 13|655+2 · 4 ↔ 13|663 ↔ 13|66+3 · 4 ↔ 13|78 ↔ 13|7+8 · 4 ↔ 13|39 (true)

Proof:
We have:
$N = 10^n a_n + 10^{n-1}a_{n-1} + 10^{n-2}a_{n-2} + \cdots + 10^2 a_2 + 10a_1 + a_0$
Let:
$N = 10^n a_n + 10^{n-1}a_{n-1} + 10^{n-2}a_{n-2} + \cdots + 10^2 a_2 + 10a_1 + a_0 = 13k$

for some integer k. Then:

$$N = 10(10^{n-1}a_n + 10^{n-2}a_{n-1} + 10^{n-3}a_{n-2} \dots + 10a_2 + a_1) + a_0 = 13k$$

Add $40a_0$ and substract on the equation to get:

$$N = 10(10^{n-1}a_n + 10^{n-2}a_{n-1} + 10^{n-3}a_{n-2} \dots + 10a_2 + a_1) + 40a_0 - 40a_0 + a_0$$
$$= 10(10^{n-1}a_n + 10^{n-2}a_{n-1} + 10^{n-3}a_{n-2} \dots + 10a_2 + a_1 + 4a_0) - 39a_0 = 13k$$

$$13k + 39a_0 = 10(10^{n-1}a_n + 10^{n-2}a_{n-1} + 10^{n-3}a_{n-2} \dots + 10a_2 + a_1 + 4a_0) \equiv 0 \pmod{13}$$

This is equivalent with:

$$10(\overline{a_n a_{n-1} a_{n-2} \dots a_2 a_1} + 4a_0) \equiv 0 \pmod{13}$$

Therefore, since 10$\equiv 10 \pmod{13}$, for N to be divisible by 13, it must be true that (3)

$$\overline{a_n a_{n-1} a_{n-2} \dots a_2 a_1} + 4a_0 \equiv 0 \pmod{13}$$

3. **Divisibility criterion with 7, 11, 13**

Rule:
A natural number is divisible by 7 (or 11, or 13) if and only if the difference between the two natural numbers obtained by "cutting" the given number so that the right remains a 3-digit number is divisible by 7 (or 11, or 13).

Proof:

Let m=$\overline{a_n a_{n-1} \dots a_1 a_0}$, n∈ ℕ , n≥2 and p=$\overline{a_n a_{n-1} \dots a_3}$, q=$\overline{a_2 a_1 a_0}$. Then m=$10^3$·p+q=(7·11·13-1)·p+(q-p). It follows that 7 | m if and only if 7 | (q-p).

Example:
If the number is 6943104.
6943-104=6839
839-6=833. (833=7·119 ⇒7|6943104)

4. **Divisibility criterion with 7, 19**

A natural number is divisible by 7 (or 19) if and only if the sum of the last two digits magnified 4 times and the number of the other digits is divisible by 7 (or 19).

Proof:
Let m=$\overline{a_n a_{n-1} \dots a_1 a_0}$, n∈ ℕ ,n≥2 and p=$\overline{a_n a_{n-1} \dots a_2}$, q=$\overline{a_1 a_0}$. Then 4m=4·$10^2$·p+4·q=(3·7·19+1)·p+(p+4·q). It follows that 7 | m if and only if 7 | (p + 4 · q). (4)

Example:
If the number is 1110987.
11109+4·87=11457
114+4·57=342
3+4·42=171. (171=7·24+3 ⇒ 7 does not divide 1110987)

Notes d'édition
(1) L'argument utilisé pour supprimer le 10, à savoir que 10 est congru à 3 modulo 7, n'éclaire en rien. Ce qui est important est que, puisque 10 et 7 (ou 3 et 7 si on a réduit modulo 7) sont premiers entre eux, 10 (ou 3) est inversible modulo 7, permettant ainsi de le « supprimer » du membre de gauche en multipliant par son inverse.
(2) Curieusement l'énoncé ne se présente pas comme un critère (une CNS). Pourtant il en est bien un, tout comme le précédent.
(3) Même remarque qu'au (1).
(4) L'énoncé est ambigu. Il faut bien sûr entendre le « and » comme la somme.