Le sujet dont vous êtes l'auteur

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Introduction:

You probably know what a prime number is. These numbers fascinated the mathematicians, especially Euclid. There are a lot of questions about them, which seem simple to solve at the beginning, but some of them haven't been solved by mathematicians for centuries. I suggest you ask yourself a subject, which you think is interesting, which you do not already know the answer to, that you had never encountered before and on which you can start by doing simulations with numbers. It's your turn.

The first problem

Description:

Let it be p, a natural prime number. There is a table of values (an array) with n lines and n columns with integers. We have to find out in how many ways we can fill this array, so that the product of the elements from every line and column is equal to $+p$ or $-p$.

The Beginning:

We started by doing some simulations. Let's give more examples, when n is equal to 2 or 3 and write the possibilities:

First example:

For n = 2:

$\pm$p	$\pm$1
$\pm$1	$\pm$p

$\pm$1	$\pm$p
$\pm$p	$\pm$1

There are only 2 possibilities to build the $2\ x\ 2$ array. As you can see, the product of the elements from every line or column is $\pm p$. The total number of possibilities is: $2^4 + 2^4 = 32$

Second example:

For n = 3:

$\pm p$	± 1	± 1
± 1	$\pm p$	± 1
± 1	± 1	$\pm p$

$\pm p$	± 1	± 1
± 1	± 1	$\pm p$
± 1	$\pm p$	± 1

± 1	$\pm p$	± 1
$\pm p$	± 1	± 1
± 1	± 1	$\pm p$

± 1	$\pm p$	± 1
± 1	± 1	$\pm p$
$\pm p$	± 1	± 1

± 1	± 1	$\pm p$
$\pm p$	± 1	± 1
± 1	$\pm p$	± 1

± 1	± 1	$\pm p$
± 1	$\pm p$	± 1
$\pm p$	± 1	± 1

We can build 6 tables of values where the 3 positions of ±p differ from each other. For each array we have 2^9 possibilities to fill it. In conclusion, for n = 3, we get a total of $2^9 \cdot 6$ ways to write the table of values.

Demonstration:

From the examples, we can realise that there are 2 very important things to think about:

- For a line to have the product of the elements equal to $+p$ (or $-p$), then one element from the line must be equal to $+p$ (or $-p$) and all of the other ones from the same line must be equal to $+1$ (or -1)

- For a column to have the product of the elements equal to $+p$ (or $-p$), then we have to be very careful how to arrange the elements equal to $+p$ (or $-p$) from the lines above.

Let's calculate in how many ways we can arrange the elements of $+p$ (or $-p$), which we will multiply with the number of possibilities where we arrange the elements of $+1$ (or -1).

For the first line, we have n places where we can put $+p$ (or $-p$), so $2 \cdot n$ possibilities.

For example, let's place $\pm p$ on the second column:

± 1	$\pm p$	± 1	± 1	....	± 1
…………	……………	……………	……………	....	……………

For the second line, we have only $(n-1)$ places left to put $+p$ (or $-p$), so $2 \cdot (n-1)$ possibilities, because of the fact that we can't place $+p$ (or $-p$) on the same column as we placed it on the first line, which is column number 2 in our example.

For example, let's place $+p$ (or $-p$) on the third column:

±1	$\pm p$	±1	±1	……………	±1
±1	±1	$\pm p$	±1	……………	±1
……………	……………	……………	……………	……………	....

For the third line, we have only $(n-2)$ places left to put $+p$ (or -p), so $2 \cdot (n-2)$ possibilities, because of the fact that we can't place $+p$ (or $-p$) on the same column as we placed it on the first two lines, which are columns 2 and 3 in our example.

For example, let's place $+p$ (or $-p$) on the fourth column.

±1	$\pm$p	±1	±1	…………	±1
±1	±1	$\pm$p	±1	………..	±1
±1	±1	±1	$\pm$p	…………	±1
…………	…………	………	…………	…………	....

We will continue, based on the same algorithm, until we will be on the n[th] line. We have only 1 place left to put $+p$ (or $-p$), so $2 \cdot 1$ possibilities, because of the fact that we can't place $+p$ (or $-p$) on the same column as we placed it on the first $(n-1)$ lines.

±1	$\pm$p	±1	±1	±1	……………
±1	±1	$\pm$p	±1	±1	……………
±1	±1	±1	$\pm$p	±1	……………
±1	±1	±1	±1	$\pm$p	……………
……………	……………	……………	……………	……………	……………
……………	……………	……………	……………	……………	……………

Now, we have to make the product of every number of possibilities from all of the lines. Therefore, the number of ways we can place $+p$ (or $-p$) in the table of values is:
$(2 \cdot n) \cdot [2 \cdot (n-1)] \cdot [2 \cdot (n-2)] \cdot \ldots \cdot (2 \cdot 1) = 2^n \cdot n!$, $n \geq 1$, this is **relation number I**

Now, we have to find a way to count the number of possibilities to place $+1$ (or -1) in the table of values. Let's see in how many ways we can place +1 (or -1) in a line with n integers.

± 1	± 1	± 1	………	$\pm$p	………	± 1	± 1
↓ 2 ways	↓ 2 ways	↓ 2 ways				↓ 2 ways	↓ 2 ways

- we have $2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^{n-1}$ possibilities to fill a line
- there are n lines in the table of values

In total, we have $2^{n-1} \cdot 2^{n-1} \cdot \ldots \cdot 2^{n-1} = 2^{n(n-1)}$ possibilities to place ± 1 in the array, $n \geq 1$, this is **relation number II**

Final Answer

From multiplying equations I and II, we will have that the final answer is:

$$\mathbf{2^n \cdot n! \cdot 2^{n(n-1)} = 2^{n^2} \cdot n!, n \geq 1}$$

The second problem

Description:

A natural number N is the product of n distinct prime numbers. In how many distinct ways can this number be represented as the difference of two nonzero perfect squares ?

The Beginning:

Let $N = p_1 \cdot p_2 \cdot \ldots \cdot p_n$, where $p_1, p_2, \ldots, p_n$ are distinct prime numbers

We find $x, y \in \mathbb{N}^*$, so that $N = x^2 - y^2 \Leftrightarrow N = (x - y)(x + y)$

We started by doing some simulations. Let's give more examples, when n is equal to 2 (or 3) and we chose 2 (or 3) random prime numbers, then we write the possibilities:

First example:

For $n = 2$

- If we choose 3 and 5 as our prime numbers $\Rightarrow N = 15$
  - $x + y = 5$ and $x - y = 3 \Rightarrow 2x = 8 \Longrightarrow x = 4$ and $y = 1$

  - $x + y = 15$ and $x - y = 1. \Rightarrow 2x = 16 \Longrightarrow x = 8$ and $y = 7$

  As shown we have 2 possibilities.
- If we chose 2 and 3 as our prime numbers $\Rightarrow N = 6$
  - $x + y = 3$ and $x - y = 2 \Rightarrow 2x = 5 \Rightarrow x \notin \mathbb{N}$
  - $x + y = 6$ and $x - y = 1 \Rightarrow 2x = 7 \Rightarrow x \notin \mathbb{N}$

  In this case we have zero possibilities

Second example:

For $n = 3$

- If we choose 3, 5 and 7 as our prime numbers $\Rightarrow N = 105$
  - $x + y = 15$ and $x - y = 7 \Rightarrow 2x = 22 \Longrightarrow x = 11$ and $y = 4$
  - $x + y = 21$ and $x - y = 5 \Rightarrow 2x = 26 \Longrightarrow x = 13$ and $y = 8$
  - $x + y = 35$ and $x - y = 3 \Rightarrow 2x = 38 \Longrightarrow x = 19$ and $y = 16$
  - $x + y = 105$ and $x - y = 1 \Rightarrow 2x = 8 \Longrightarrow x = 53$ and $y = 52$

  As shown we have 4 possibilities.

- If we chose 2, 3 and 5 as our prime numbers $\Rightarrow N = 30$
  - $x + y = 6$ and $x - y = 5 \Rightarrow 2x = 11 \Rightarrow x \notin \mathbb{N}$
  - $x + y = 10$ and $x - y = 3 \Rightarrow 2x = 13 \Rightarrow x \notin \mathbb{N}$
  - $x + y = 15$ and $x - y = 2 \Rightarrow 2x = 17 \Rightarrow x \notin \mathbb{N}$
  - $x + y = 30$ and $x - y = 1 \Rightarrow 2x = 31 \Rightarrow x \notin \mathbb{N}$

  In this case we have zero possibilities.

Demonstration:

$N = x^2 - y^2 = p_1 \cdot p_2 \cdot p_3 \cdot .... \cdot p_n \Leftrightarrow (x - y) \cdot (x + y) = p_1 \cdot p_2 \cdot p_3 \cdot .... \cdot p_n$

If one of the prime numbers, $p_1, p_2, ..., p_n$ is equal to 2, then $(x + y)$ is even and $(x - y)$ is odd or $(x + y)$ is odd and $(x - y)$ is even. From these cases, we obtain that $(x + y) + (x - y)$ is odd $\Rightarrow 2x$ is odd $\Rightarrow x \notin \mathbb{N}$.

Therefore, there are no solutions when one of the numbers $p_1, p_2, ..., p_n$ is 2. This means that $p_1, p_2, ..., p_n$ are distinct and odd prime numbers.

Therefore:

$p_1 \mid (x + y)$ or $p_1 \mid (x - y) \rightarrow$ 2 possibilities
$p_2 \mid (x + y)$ or $p_2 \mid (x - y) \rightarrow$ 2 possibilities
$p_3 \mid (x + y)$ or $p_3 \mid (x - y) \rightarrow$ 2 possibilities
....
$p_n \mid (x + y)$ or $p_n \mid (x - y) \rightarrow$ 2 possibilities

We have 2 possibilities for each prime number. Therefore, we have $2 \cdot 2 \cdot 2 \cdot ... \cdot 2 = 2^n$ possibilities in total.

Also we know that the sum between 2 distinct numbers is greater or equal with their difference, $x + y \geq x - y$.
If $x + y = x - y \Rightarrow 2y = 0 \Rightarrow y = 0$ $(false) \Rightarrow x + y > x - y$.
From the fact that $x + y > x - y$, we realised that the number of possibilities is cut in half $\Rightarrow$ we will obtain $2^n : 2 = 2^{n-1}$ possibilities

Final answer:

We have 2^{n-1} possibilities to write a natural number formed as a product of distinct prime numbers, as a difference of 2 squares.

C++ Program

The problem is very difficult to solve, when n is big. Of course, technology can help us a lot and that's why we made a C++ program.

You can choose how many different prime numbers you want to multiply and after that which are these.
Our program will show all the solutions $x, y \in \mathbb{N}^*$, where $x^2 - y^2 = p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_n$ and the number of possibilities.

```cpp
#include <iostream>
#include <fstream>
#include <array>
#include <vector>
using namespace std;

int x, cnt;

void f(int target) {

    for (int diff = 1; diff * diff <= target; ++diff) {
        if (target % diff != 0)
            continue;

        int sum = target / diff;

        if (sum % 2 != diff % 2)
            continue;

        int b = (sum - diff) / 2;
        int a = sum - b;

        cnt++;
        std::printf("%d %d\n", a, b);

    }
}

void case_1(){
    int randVal, p = 1;
    cout << "Chose how many numbers you want to multiply: ";
    cin >> randVal;
    cout << '\n';
    cout << "The numbers are: ";
    while(randVal != 0){
        cin >> x;
        p *= x;
        randVal--;
    }
    cout << '\n';
    cout << "These are the results: " << "\n";
    f(p);
    cout << '\n';
    cout << "The number of posibilities: ";
    cout << cnt << '\n';
}

int main() {
    case_1();
    return 0;
}
```

If $n = 5$ and $p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, the program shows the 16 solutions in less than a second. It would be too hard to find all the solutions without the help of the program.

```
Chose how many numbers you want to multiply:5

The numbers are:3 5 7 11 13

These are the results:
7508 7507
2504 2501
1504 1499
1076 1069
688 677
584 571
508 493
368 347
244 211
232 197
212 173
164 109
148 83
136 59
128 37
124 19

The number of posibilities: 16
```