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Modeling of Plant Growth

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Contents

1 The Task

Modeling of plant growth:
Study tree leaves, flowers or snail shells to come up with a L-system model of evolution.

2 Solution

2.1 What are L-Systems?

In 1968, Hungarian botanist Aristid Lindenmayer developed a grammar-based system to model the growth patterns of plants. Lindenmayer systems — or L-systems for short — were originally conceived as a mathematical theory of plant development. However, we are going to use them generate self-similar fractal patterns.[1]

At the beginning, L- systems were defined as linear arrays of finite automata; later, however, they were reformulated into the more suitable framework of grammar-like constructs. From then on, the theory of L- systems was developed essentially as a branch of formal language theory.[2]

The essential feature about L-systems is that the rewriting of a string happens in a parallel manner, contrary to the sequential rewriting in grammars. This means that at every step of the rewriting process according to an L-system every letter has to be rewritten.[3]

2.2 The Original L-System

Lindenmayer's original L-system was for modelling the growth of algae.

The variables used are A and B and there are no constants. The rules are simple: every A transforms into AB ($A \longrightarrow AB$) and every B transforms into an A ($B \longrightarrow A$).

Starting from B (the axiom, where n = 0, n being the number of transformations), we obtain:
$n = 0 : B$
$n = 1 : A$
$n = 2 : AB$
$n = 3 : ABA$
$n = 4 : ABAAB$
$n = 5 : ABAABABA$
$n = 6 : ABAABABAABAAB$
$n = 7 : ABAABABAABAABABAABABA$
$n = 8 : ABAABABAABAABABAABABAABAABABAABAAB$, and so on.

The result is the sequence of Fibonacci words.
If we count the length of each string, we obtain the famous Fibonacci sequence of numbers (skipping the first 1, due to our choice of axiom):
1, 2, 3, 5, 8, 13, 21, 34, 55, 89 ... Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two arrays, a_k and b_k defined by the number of A and respectively B letters in the k-th string.

a_0=1	a_1=1	a_2=2	a_3=3	a_4=5	a_5=8	a_6=13	a_7=21
b_0=0	b_1=1	b_2=1	b_3=2	b_4=3	b_5=5	b_6=8	b_7=13

We observe that $a_2 = a_1 + a_0$, $a_3 = a_2 + a_1$ and so on.

Because of the recurrence rules, where each B from step k-1 becomes an A in the k-th string, and every A from step k-1 becomes an AB in the k-th string, we have $a_k = a_{k-1} + b_{k-1}$ and $b_k = a_{k-1} \implies$

$$\begin{cases} a_k = a_{k-1} + a_{k-2} \\ b_k = a_{k-1} \end{cases}$$

Thus, $a_k = F_{k+1}$ and $b_k = F_k$, where $(F_n)_{n\geq 0}$ is the Fibonacci Sequence.

2.3 Fibonacci in Nature

The Fibonacci sequence is one of the most famous results in mathematics. Each number in the sequence is the sum of the two numbers before it.
So, the sequence goes:
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, and so on.

The mathematical equation describing it is $x_{n+2} = x_{n+1} + x_n, \forall n \in \mathbb{N}$, where $x_0 = 0, x_1 = 1$.

It's been called "nature's secret code," and "nature's universal rule." It is said to govern the dimensions of everything from the Great Pyramid at Giza, to seashells and snail shells.

Observing the geometry of plants, flowers or fruit, it is easy to recognize the presence of recurrent structures and forms. The Fibonacci sequence, for example, plays a vital role in the arrangement of leaves, branches, flowers or seeds in plants, with the main aim of highlighting the existence of regular patterns.

Let us insert an image of the Fibonacci spiral. By simply drawing adjacent squares with the side length of the numbers from the sequel, and then by contouring a curve line through 2 opposite vertices of each square, the Fibonacci spiral is formed.
As we can see, it looks very similar to snail shells.

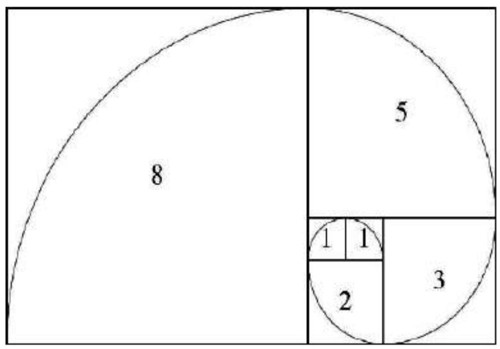

Figure 1 – Fibonacci Spiral

Figure 2 – Snail Shell

We can easily find the numbers of the Fibonacci sequence in flowers such as daisies or sunflowers, where the pollen seeds are arranged in overlapping spirals, but also in cauliflower and broccoli.

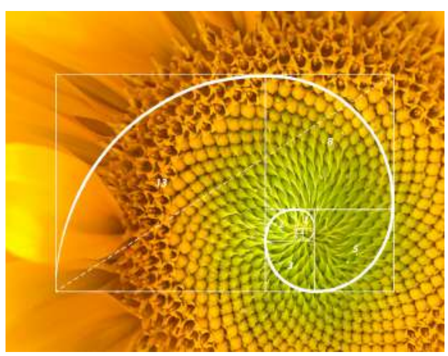

Figure 3 – Fibonacci in Sunflowers

Figure 4 – Fibonacci in other Flowers

Moreover, it is usual for many trees to grow their branches in a way that respects the Fibonacci sequence. We can see in the picture below that every higher level has a number of branches equal to the sum of the previous two number of branches (from the last two height levels).

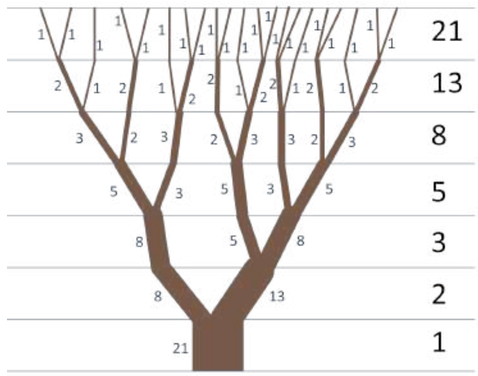

Figure 5 - Evolution of Tree Branches

2.4 L-Systems in Nature

A central concept of L-systems is rewriting: successively replacing parts of a simple initial object using a set of rewriting rules. Another fundamental concept is recursion, which is the repeated application of a rule to successive results. Therefore, the rewriting concept of L-systems is an efficient mechanism to apply recursion in order to achieve complex fractals. [4]

2.4.1 Model 1

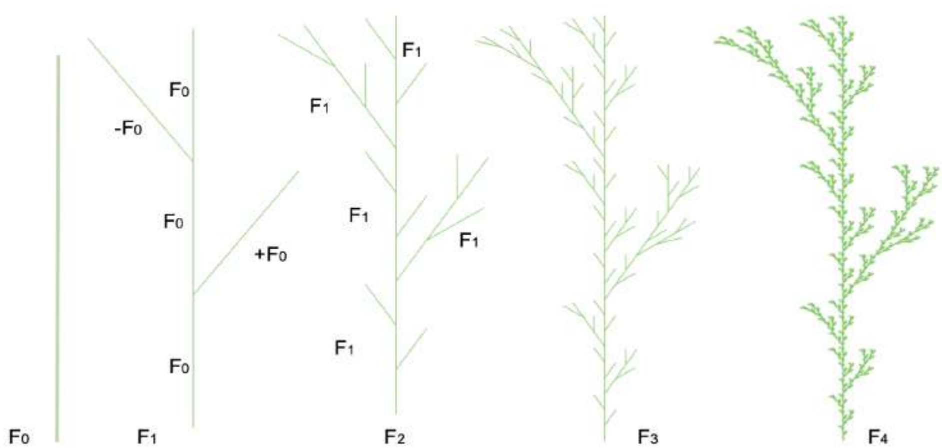

Figure 6 - Model 1 Evolution

(For visibility reasons, all five models in Figure 6 have the same dimension, even though, in reality every step makes the model three times taller than the previous one)

Let the initiator be F_0 and F_n the model obtained after n transformations. When writing a recurrence relation, we define the following three operators:

1) F, which means drawing the shape as it is (the shape being basically the previous model);

2) $+$, which means drawing the previous model on the right side;

3) $-$, which means drawing the previous model on the left side.

There are also square brackets when writing the recurrence relation. When the square bracket closes, the next segment drawn would have the same orientation as the one before the square bracket.

The rule for this example is:

$$F_{n+1} = F_n[+F_n]F_n[-F_n]F_n$$

That means that F_{n+1} is formed by drawing F_n vertically, then drawing to the right, then vertically again then to the left and finally, vertically at the end.

Hence, F_n is repeated 5 times in F_{n+1}, as we said before: 3 times vertically and 2 times in lateral branches.

To count the smallest lateral branches of the plant model at each step, we define B_n as their number on step n.

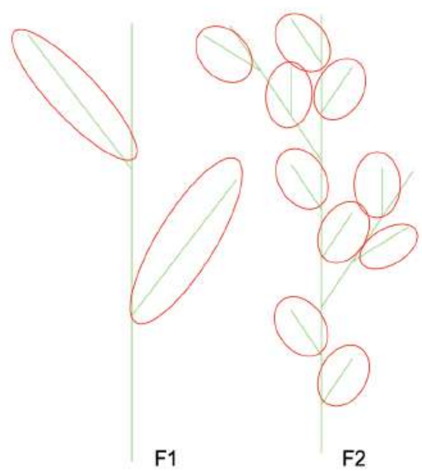

Figure 7 - Smallest Lateral Branches

As we can see in Figure 7, where the smallest lateral branches are circled in red, for F_1 we have $B_1 = 2$. For F_2, we have 10 small lateral branches, so B_2 is 10. For a larger n, a formula is needed in order to find the correct number and it is easy to prove by induction that

$$B_n = 2 \cdot 5^{n-1}$$

Given that F_{n+1} is formed by using F_n 5 times, for every F_n used, we get a number B_n of buds. Thus, in F_{n+1}, the number B_{n+1} will be $5 \cdot B_n$. The induction proof is now easier to be done.

We have $P(n) : B_n = 2 \cdot 5^{n-1}, \forall n \geq 1$, with $P(1)$ and $P(2)$ being true, as previously presented. The last step is to prove that if $P(k)$ stands, then $P(k+1)$ stands, too.

$P(k) : B_k = 2 \cdot 5^{k-1}$ and $B_{k+1} = 5 \cdot B_k \implies B_{k+1} = 2 \cdot 5^k$. Therefore, $P(k+1)$ is true and the induction is complete.

To sum up all of the above, we can say that for the first model, the number of smallest lateral branches for a step n is $B_n = 2 \cdot 5^{n-1}$.

2.4.2 Model 2

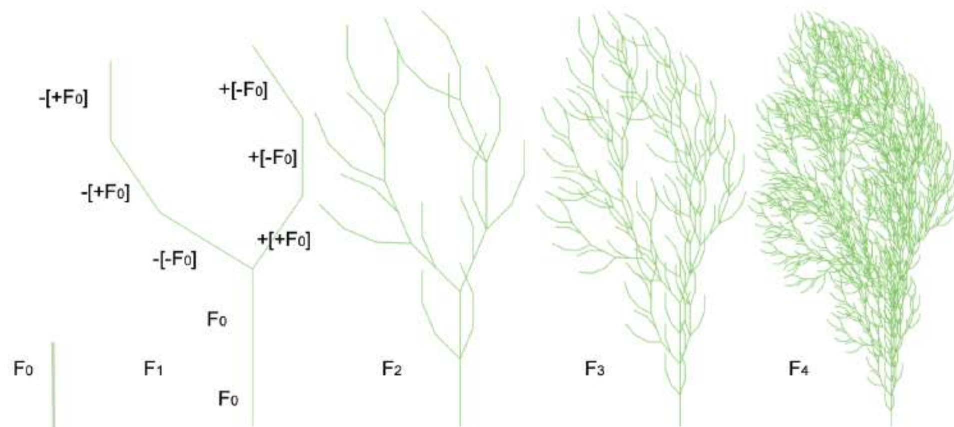

Figure 8 - Model 2 Evolution

(For visibility reasons, the last four models in the Figure 8 have the same dimension, even though, in reality every step makes the model much taller than the previous one. The evolution from the initiator F_0 to F_1 is, in scale, true to reality)

For the second model, the rule is slightly more complicated, but the operators are the same: $+$ means going right and $-$ means going left.

The recurrence rule is:

$$F_{n+1} = F_n F_n - [-F_n + F_n + F_n] + [+F_n - F_n - F_n]$$

This means that F_{n+1} is formed by drawing F_n two times vertically, then, it splits into two principal branches. On the left hand side branch, we draw F_n to the left, then two times to the right and we come back to the point where the square bracket opened, i.e. the point of bifurcation into the two branches. On the right hand side principal branch we draw F_n to the right, then to the left 2 times, according to the rule.

Therefore, F_{n+1} has 8 similar parts, each of them representing the previous model, that is F_n.

To count the number of buds of the plant model at each step, we define B_n as their number on any step n.

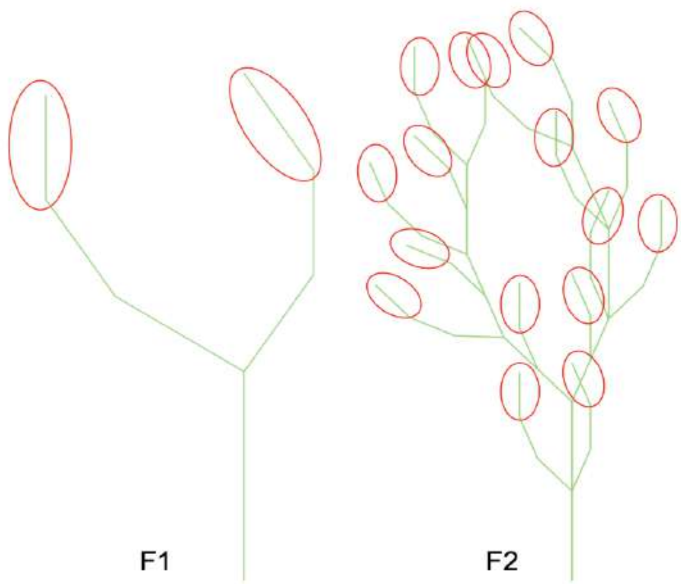

Figure 9 – Number of Buds

As presented in Figure 9, the ends circled in red representing the buds of F_1 and F_2, we observe that $B_1 = 2$ and $B_2 = 16$.

Knowing that F_{n+1} is formed by drawing F_n eight times, one can state that

$$B_{n+1} = 8 \cdot B_n$$

.

Let us prove using mathematical induction that

$$B_n = 2^{3n-2}, \forall n \geq 1$$

While $P(1) : B_1 = 2^1$ and $P(2) = 2^4$ are correct, the only thing left to prove is that the truth of $P(k)$ implies that $P(k+1)$ also stands.

From $B_k = 2^{3k-2}$ and $B_{k+1} = 8 \cdot B_k$, we obtain that $B_{k+1} = 8 \cdot 2^{3k-2} \Longleftrightarrow B_{k+1} = 2^{3k+1} \Longleftrightarrow B_{k+1} = 2^{3(k+1)-2}$. With this, the induction is complete.

So, we can say that for the second model, the number of buds for a step n is $B_n = 2^{3n-2}$.

3 Conclusion

Lindenmayer's original L-system was for modelling the growth of algae, especially the red algae. The growth models of Figure 6 and Figure 8 are quite similar to the ones found in nature. They might not be just the same, as the rule applied in order to model plant growth may be slightly more complicated (the operators are the same, the concepts of rewriting and recursion are still visible) and in reality, plants develop all sorts of anomalies based on external factors such as climate, soil or environment.

4 References

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Figure 2: easternleaf.com
Figure 3: clevelanddesings.com
Figure 4: pinterest.com
Figure 5: pinterest.com

Figures 6, 7, 8 and 9 were drawn by us.

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