A journey through the wonders of Fermat's Point

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ISISS "M. Casagrande", Pieve di Soligo, Treviso – Italy

Abstract

The Fermat point is a remarkable point of a triangle introduced by the French mathematician Pierre de Fermat in the XVII century. In this article we studied some known properties of such point, also managing to discover some new interesting characteristics. We started analysing triangles, moving to quadrilaterals and finally to polygons. Regarding triangles, by slightly modifying the definition of Fermat point, we obtained a new set of points and we proved that the Fermat point belongs to this set as well as the triangle centroid and the orthocenter. Moving away from triangles, we obtained some results involving regular polygons, but because of the complexity of the task we were not able to find a method to determine the Fermat point in irregular polygons. For this reason, we used calculus to develop an algorithm to approximate Fermat point.

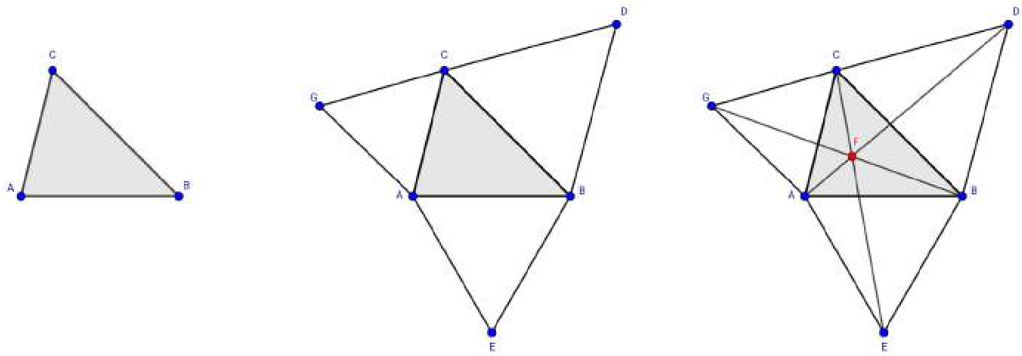
Contents

Chapter 1

Triangles

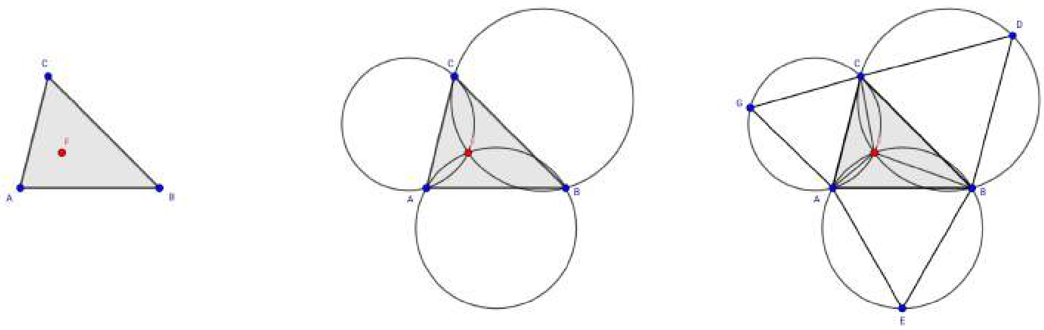
Let's start by analyzing the triangle and the definition of Fermat Point of a triangle.

Definition 1. *Considering the equilateral triangles constructed on the sides of a triangle, the Fermat point is the intersection of the segments whose endpoints are a vertex of the triangle and the vertex that doesn't belong to the triangle of the equilateral triangle built on the opposite side.*

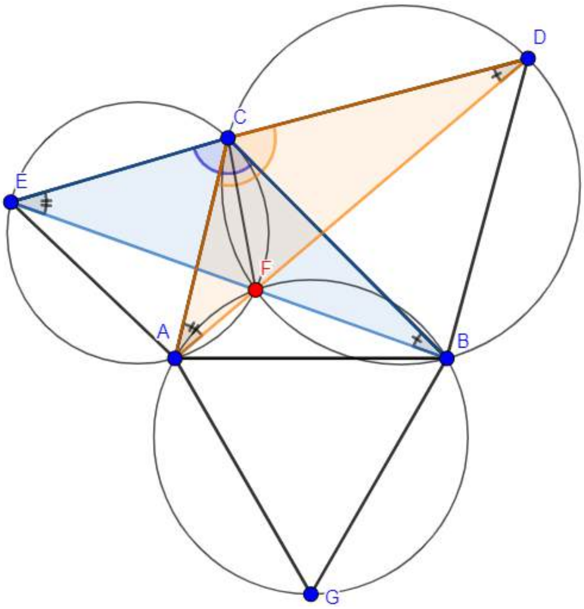


We can now obtain the following results:

Theorem 1. *The Fermat point is the intersection of the circumferences circumscribed to the equilateral triangles we mentioned before.*



Proof. Let us consider the equilateral triangles ACE, BCD and ABG built on the sides of the triangle ABC. We can prove that the triangles EBC and ADC are congruent for the first congruence criterion, indeed $AC \cong EC$ and $CD \cong BC$ for construction and $E\widehat{C}B \cong A\widehat{C}D$ due to the sum of congruent angles ($A\widehat{C}B$ in common $+ 60°$). In particular $C\widehat{E}B \cong C\widehat{A}D$, so A and E see the segment CF under the same angle. Therefore the quadrilateral $CFAE$ is inscribable in a circumference. In the same way we can prove that the quadrilateral $CFBD$ is inscribable in a circumference. Particularly $C\widehat{F}A \cong C\widehat{F}B \cong 120°$ (in inscribed quadrilaterals the opposite angles are supplementary). Consequently, also $A\widehat{F}B$ measures $120°$ and the quadrilateral $AGBF$ is inscribable in a circumference. In this way we have proved that the three circumferences meet each others in the Fermat point F.



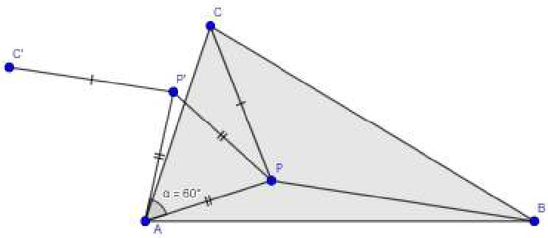
□

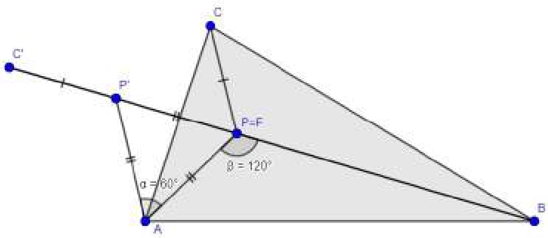
Theorem 2. *In a triangle, the segments whose endpoints are the Fermat point and a vertex of the triangle forms angles of* $120°$.

Proof. This theorem is a consequence of what we have seen in the proof of the theorem 1. □

Theorem 3. *In a triangle, the Fermat point minimizes the sum of the distances from the three vertices* A, B *and* C.

Proof. Let's consider a point P inside a triangle ABC. Let's rotate with center A the polyline composed by AP and PC by an angle of $60°$. We obtain a segment $AP' \cong AP$ and a segment $C'P' \cong CP$. The triangle APP' is equilateral. So also $P'P \cong PA$. Therefore the sum $AP + BP + CP$ is equivalent to the sum $C'P' + P'P + PB$. This sum is minimum when the points C', P', P and B are aligned. In this case the angle $A\widehat{P}B = 120°$. In conclusion the sum is minimum when the point P coincides with the Fermat point F.





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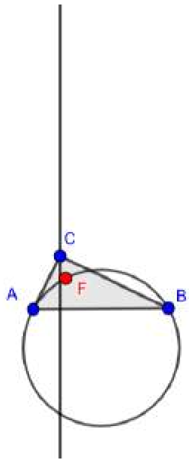
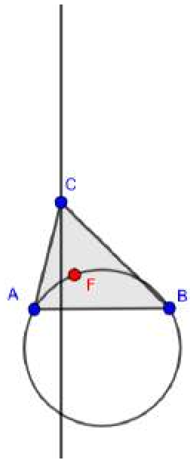
1.1 Some particular cases

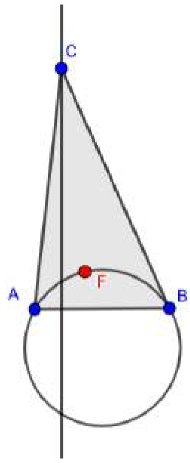
Now let's study some properties of the Fermat point in particular triangles.

1.1.1 C belongs to a line perpendicular to the side AB

Theorem 4. *In a triangle ABC, if the vertex C lies on a line perpendicular to the side AB, the Fermat point of ABC belongs to the arc of circumference circumscribed to the equilateral triangle built on the side AB.*





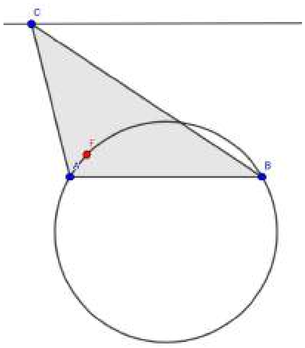


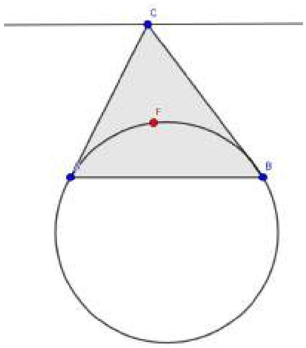
Proof. As we said before, the angle AFB measures $120°$, so the quadrilateral $AFBD$, where D is the third point of the equalateral triangle built on AB, is circumscribed to a circumference and the point F belongs to this. □

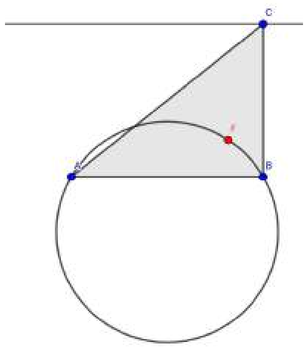
1.1.2 C belongs to a line parallel to the side AB

Following the same reasoning we can obtain this theorem:

Theorem 5. *In a triangle ABC, if the vertex C lies on a line parallel to the side AB, the Fermat point belongs to the arc of circumference circumscribed to the equilateral triangle built on the side AB.*



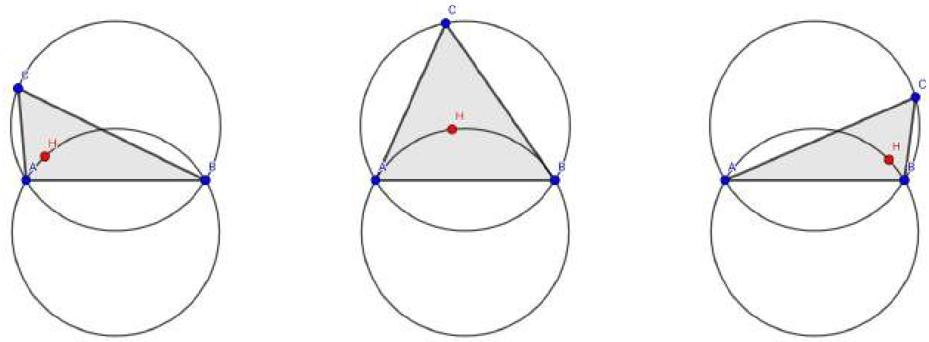




1.1.3 C belongs to the circumference passing through A and B

Following again the same reasoning we can obtain this theorem:

Theorem 6. *In a triangle ABC, if the vertex C lies on the circumference passing through A and B, the Fermat point belongs to an arc of the circumference circumscribed to the equilateral triangle built on the base AB.*



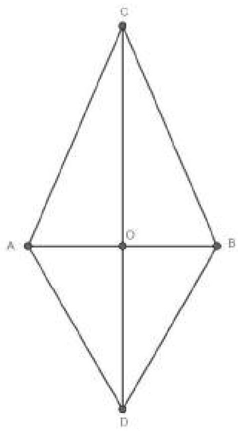
1.2 Isosceles triangle

1.2.1 C belongs to the line perpendicular to the base AB passing through its midpoint in an isosceles triangle

First we prove the following theorem:

Theorem 7. *In an isosceles triangle the Fermat point belongs to the axis of AB.*

Proof. Considering the triangles ACD and BCD they are congruent because $BC \cong AC$, $AD \cong DB$ and DC in common. So $A\widehat{D}O \cong B\widehat{D}O$. Considering ADO and BDO, they have $A\widehat{D}O \cong B\widehat{D}O$, $AD \cong DB$, OD in common. Then $D\widehat{O}B \cong D\widehat{O}A \cong 90°$. Due to its definition the Fermat point belongs to the segment DC that is the axis of AB.



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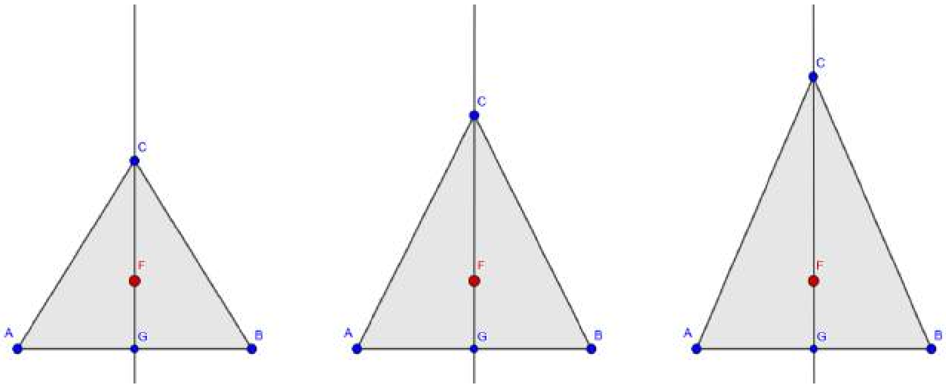
Now we obtain:

Theorem 8. *In a triangle ABC, if the point C belongs to the axis of AB, the Fermat point is always the same.*



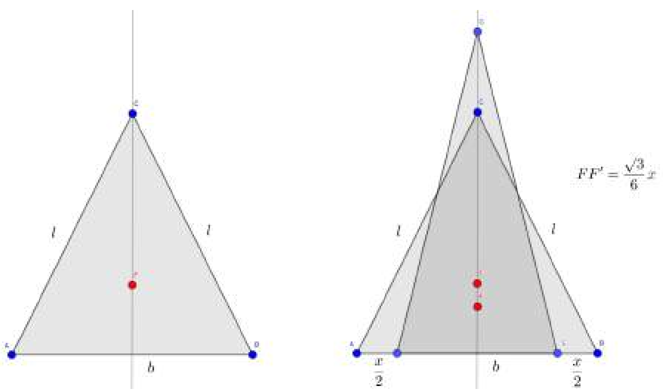
Proof. Let's consider the base $AB = b$. We know the Fermat point F belongs to the axis of AB, so the angles FAG and FBG measure 30°, indeed AFB is 120°, as we have seen before. AG is $\frac{b}{2}$ so:

- $AF = \dfrac{AG}{\sin(60°)} = \dfrac{b/2}{\sqrt{3}/2} = \dfrac{b}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}b$
- $FG = AF \cdot \sin(30°) = \dfrac{\sqrt{3}}{3}b \cdot \dfrac{1}{2} = \dfrac{\sqrt{3}}{6}b$

Therefore for any C belonging to the axis of AB, F belongs to the axis as well and $FG = \dfrac{\sqrt{3}}{6}b$ □

1.2.2 The base of an isosceles triangle restricts

Theorem 9. *In an isosceles triangle, with sides of length l and base b, restricting the base from b to b - x, the Fermat point is lowered by $\dfrac{\sqrt{3}}{6}x$*



Proof. Considering the triangle FAB the height relative to the base measures $\dfrac{l}{2}$.
Instead in the triangles $F'DE$ the length of the segment is $\sqrt{3}l - x$.
We know that :

- $AH = \frac{\sqrt{3}}{2}l - \frac{x}{2}$
- $F'H = \tan(30°) \cdot AH \to F'H = \frac{1}{2}l - \frac{\sqrt{3}}{6}x$

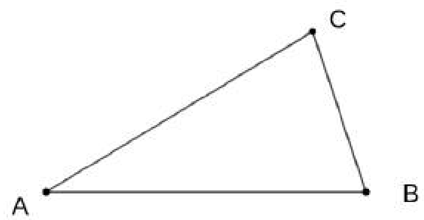
In conclusion $FF' = \frac{l}{2} - \left(\frac{1}{2}l - \frac{\sqrt{3}}{6}x\right) = \frac{\sqrt{3}}{6}x$ □

1.3 Cartesian coordinates of Fermat point

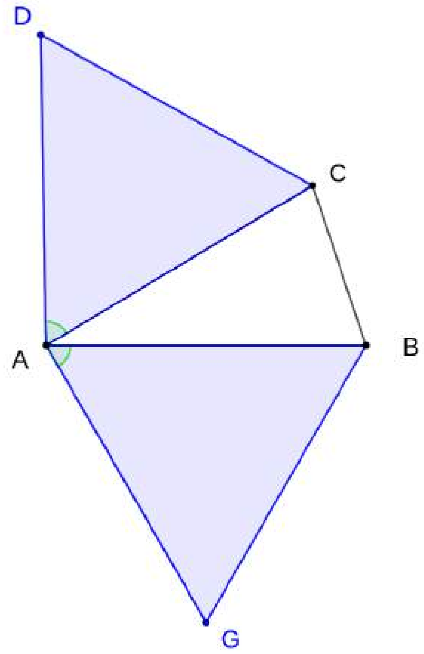
We propose two distinct methods to find the Cartesian coordinates of the Fermat point. Without losing generality, we will position the origin of the Cartesian plane at point A of the triangle and the x-axis along the side AB of the triangle.

1.3.1 First method

Let's consider a generic triangle ABC with all angles less than a right angle. The three vertices of the triangle in the coodinates system are $A(0;0), B(x_B;0)$ and $C(x_C;y_C)$ with $x_B, x_C, y_C \in \mathbb{R}$.



To find the Fermat point F we can build the equilateral triangles on AC and AB.



Therefore we can find the coordinates of the points D and G by rotating respectively C and B by an angle of $\frac{\pi}{3}$. We can calculate the coordinates of the two new points using the *matrix of rotation*:

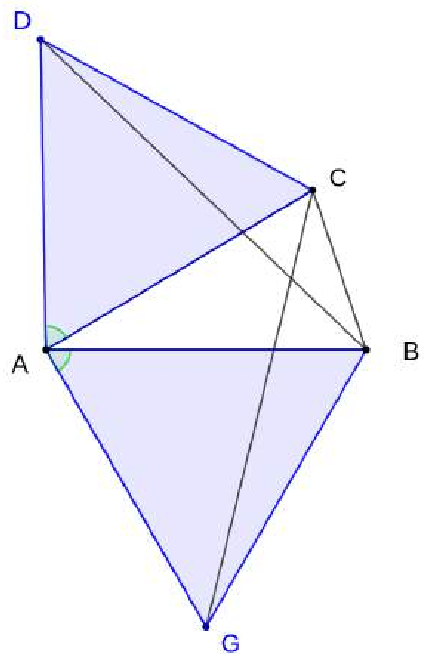
$$R_\varphi = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

The two points themselves will indeed be:

$$D = \begin{bmatrix} x_D \\ y_D \end{bmatrix} = \begin{bmatrix} \cos\left(\dfrac{\pi}{3}\right) & -\sin\left(\dfrac{\pi}{3}\right) \\ \sin\left(\dfrac{\pi}{3}\right) & \cos\left(\dfrac{\pi}{3}\right) \end{bmatrix} \begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2}x_C - \dfrac{\sqrt{3}}{2}y_C \\ \dfrac{\sqrt{3}}{2}x_C + \dfrac{1}{2}y_C \end{bmatrix}$$

$$G = \begin{bmatrix} x_G \\ y_G \end{bmatrix} = \begin{bmatrix} \cos\left(\dfrac{-\pi}{3}\right) & -\sin\left(\dfrac{-\pi}{3}\right) \\ \sin\left(\dfrac{-\pi}{3}\right) & \cos\left(\dfrac{-\pi}{3}\right) \end{bmatrix} \begin{bmatrix} x_B \\ y_B \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2}x_B \\ -\dfrac{\sqrt{3}}{2}x_B \end{bmatrix}$$

We can now draw the two lines GC and DB and find their equations:



$$GC : y - y_G = m_{GC} (x - x_G)$$

$$DB : y - y_B = m_{DB} (x - x_B)$$

The two angular coefficients can be expressed as:

$$m_{GC} = \frac{\Delta y}{\Delta x} = \frac{y_C - y_G}{x_C - x_G}$$

$$m_{DB} = \frac{\Delta y}{\Delta x} = \frac{y_D}{x_D - x_B}$$

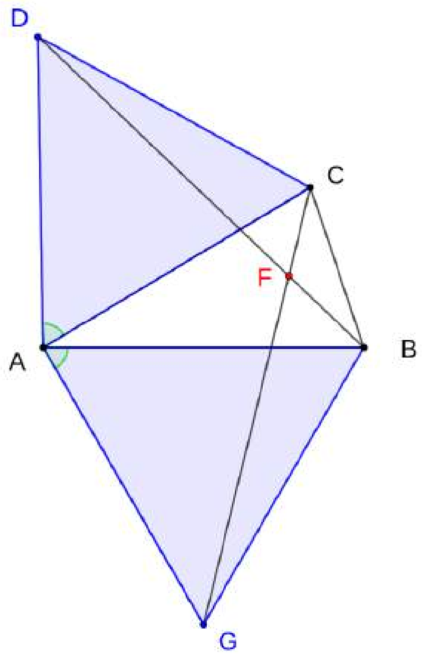
and the two equations will therefore be:

$$GC : y = \frac{y_C - y_G}{x_C - x_G} x - \frac{y_C - y_G}{x_C - x_G} x_G + y_G = sx - sx_G + y_G$$

$$DB : y = \frac{y_D}{x_D - x_B} x - \frac{y_D}{x_D - x_B} x_B = tx - tx_B$$

where $s = \dfrac{y_C - y_G}{x_C - x_G}$ and $t = \dfrac{y_D}{x_D - x_B}$. We can now find the coordinates of F by intersecting the two lines:

$$F = GC \cap DB \rightarrow \begin{cases} y = sx - sx_G + y_G \\ y = tx - tx_B \end{cases}$$



By simply solving the system, we get:

$$\begin{cases} x = \dfrac{tx_B + y_G - sx_G}{t - s} \\ y = t\dfrac{tx_B + y_G - sx_G}{t - s} - tx_B \end{cases}$$

And the Fermat point results to be the following:

$$\boldsymbol{F\left(\frac{tx_B + y_G - sx_G}{t - s}; t\frac{tx_B + y_G - sx_G}{t - s} - tx_B\right)}$$

1.3.2 Second method

The second method uses Theorem 1,that is the Fermat point is the intersection of the two circumferences circumscribed to the equilateral triangles constructed on the sides of the triangle. So we can get the Cartesian coordinates of the Fermat point by solving the following system:

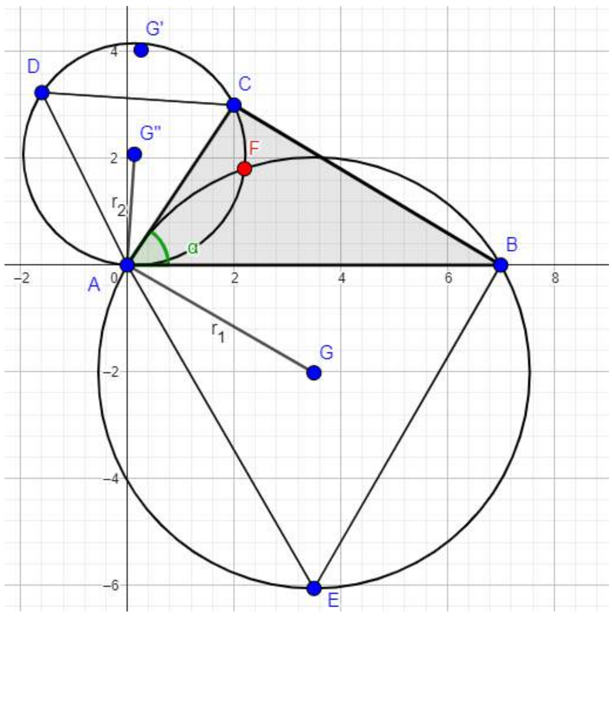
$$\begin{cases} \left[x - \dfrac{\sqrt{3}}{3}AC\sin\left(\alpha + \dfrac{2}{3}\pi\right)\right]^2 + \left[y - \dfrac{\sqrt{3}}{3}AC\sin\left(\alpha + \dfrac{\pi}{6}\right)\right]^2 = \dfrac{AC^2}{3} \\ x^2 + y^2 - xx_B + y\left(\dfrac{\sqrt{3}}{3}x_B\right) = 0 \end{cases}$$

First we find the equation of the circumference built on the basis of the triangle using the geometric center G (which coincides with the center of the circumference) and the radius (that is the distance from the point G to the point A). After that, we first rotate the circumference of an angle α and then we shrink it. In particular G has coordinates $\left(\frac{AB}{2}; -\frac{\sqrt{3}}{6}AB\right)$.

With a rotation of an angle $\alpha + \frac{\pi}{3}$ we've obtained the point
$G'\left(\frac{AB}{2}\cos\left(\alpha+\frac{\pi}{3}\right)+\frac{\sqrt{3}}{6}AB\sin\left(\alpha+\frac{\pi}{3}\right); \frac{AB}{2}\sin\left(\alpha+\frac{\pi}{3}\right)-\frac{\sqrt{3}}{6}AB\cos\left(\alpha+\frac{\pi}{3}\right)\right)$.
Thanks to the auxiliary angle method we find $G'\left(\frac{\sqrt{3}}{3}AB\sin\left(\alpha+\frac{2}{3}\pi\right); \frac{\sqrt{3}}{3}AB\sin\left(\alpha+\frac{\pi}{6}\right)\right)$.
We now observe that the ratio between X''_G and X'_G is equal to the ratio between AG'' and AG', where $AG'' = \frac{AC}{\sqrt{3}}$ and $AG' = \frac{AB}{\sqrt{3}}$; we can say the same for y''_G and y'_G.
We have obtained $G''\left(\frac{\sqrt{3}}{3}AC\sin\left(\alpha+\frac{2}{3}\pi\right); \frac{\sqrt{3}}{3}AC\sin\left(\alpha+\frac{\pi}{6}\right)\right)$. So we can find the equation of the circumnference circumscribed to the triangle ACD with center G'' and radius $r = \frac{AC}{\sqrt{3}}$:

$$\left[x-\frac{\sqrt{3}}{3}AC\sin\left(\alpha+\frac{2}{3}\pi\right)\right]^2+\left[y-\frac{\sqrt{3}}{3}AC\sin\left(\alpha+\frac{\pi}{6}\right)\right]^2=\frac{AC^2}{3}$$

the second equation can be written using the center G and the radius r_1, that is:
$x^2+y^2-xx_B+y\left(\frac{\sqrt{3}}{3}x_B\right)=0$



Example

We now give an example were we use the two method to calculate the Fermat point of the triangle ABC, with $A\,(0,0)\,, B\,(7,0)$ and $C\,(2,3)$.

First method

First of all we calculate

$$G\left(\frac{7}{2};-\frac{7\sqrt{3}}{2}\right) \qquad \text{e} \qquad D\left(1-\frac{3\sqrt{3}}{2};\sqrt{3}+\frac{3}{2}\right)$$

then

$$s=-\frac{6+7\sqrt{3}}{3} \qquad \text{e} \qquad t=-\frac{6+5\sqrt{3}}{39}$$

so the Fermat point is $F\left(\frac{273+119\sqrt{3}}{218};\frac{441+427\sqrt{3}}{654}\right)$.

Second method

Following the second method we know that $AC=\sqrt{13}$, $\alpha=\arctan\frac{3}{2}$ and $x_B=7$. The system becomes:

$$\begin{cases} \left[x-\frac{\sqrt{3}}{3}\sqrt{13}\sin\left(\arctan\frac{3}{2}+\frac{2}{3}\pi\right)\right]^2+\left[y-\frac{\sqrt{3}}{3}\sqrt{13}\sin\left(\arctan\frac{3}{2}+\frac{\pi}{6}\right)\right]^2=\frac{13}{3} \\ x^2+y^2-7x+\frac{7\sqrt{3}}{3}y=0 \end{cases}$$

We simplify it to obtain:

$$\begin{cases} \left[x-1+\frac{\sqrt{3}}{2}\right]^2+\left[y-\frac{3}{2}-\frac{\sqrt{3}}{3}\right]^2=\frac{13}{3} \\ x^2+y^2-7x+\frac{7\sqrt{3}}{3}y=0 \end{cases}$$

The solution is again the Fermat point $F\left(\frac{273+119\sqrt{3}}{218};\frac{441+427\sqrt{3}}{654}\right)$.

1.4 Sum of the distances

The aim of this section is to prove an expression that relates the sum of the distances of the Fermat point from the vertices of the triangle to the sides and the area of the triangle.

Theorem 10. *Given a triangle with sides a, b and c, with area A, the sum of the distances of the Fermat point F from the vertices $x+y+z$ is:*

$$\boldsymbol{x+y+z=\sqrt{\frac{a^2+b^2+c^2}{2}+2\sqrt{3}A}}$$

Proof. First we can compute the area of the triangle ABC as the sum of the areas of the three triangles inside ABC:

$$\begin{aligned}
A_{ABC} &= A_{AFC} + A_{AFB} + A_{FCB} \\
&= \frac{1}{2}xz\sin(AFC) + \frac{1}{2}xy\sin(AFB) + \frac{1}{2}yz\sin(BFC) \\
&= \frac{\sqrt{3}}{4}xz + \frac{\sqrt{3}}{4}xy + \frac{\sqrt{3}}{4}yz \\
&= \frac{\sqrt{3}}{4}(xy + yz + xz)
\end{aligned}$$

And therefore get to the equivalent equation:

$$xy + yz + xz = \frac{4A}{\sqrt{3}}$$

Subsequently, through the use of the cosine theorem in the three internal triangles we can calculate the sum of the squares of the three external sides of the triangle ABC. This eventually allows us to isolate the square of the sum of the three sides x, y, z:

$$a^2 = x^2 + y^2 - 2xy\cos(AFB) = x^2 + y^2 + xy$$

$$b^2 = y^2 + z^2 - 2yz\cos(CFB) = y^2 + z^2 + yz$$

$$c^2 = x^2 + z^2 - 2xz\cos(AFC) = x^2 + z^2 + xz$$

We can write the sum of the squares of the sides as:

$$\begin{aligned}
a^2 + b^2 + c^2 &= 2(x^2 + y^2 + z^2) + xy + xz + yz \\
&= 2(x^2 + y^2 + z^2) + \frac{4A}{\sqrt{3}}
\end{aligned}$$

Therefore:

$$x^2 + y^2 + z^2 = \frac{1}{2}(a^2 + b^2 + c^2) - \frac{2A}{\sqrt{3}}$$

We can now write the sum of x, y and z squared as:

$$\begin{aligned}
(x + y + z)^2 &= x^2 + y^2 + z^2 + 2(xy + yz + xz) \\
&= \frac{1}{2}(a^2 + b^2 + c^2) - \frac{2A}{\sqrt{3}} + \frac{8A}{\sqrt{3}} \\
&= \frac{1}{2}(a^2 + b^2 + c^2) + \frac{6A}{\sqrt{3}}
\end{aligned}$$

We can finally take the square roots of both sides of the equation and conclude that:

$$\boldsymbol{x + y + z = \sqrt{\frac{a^2 + b^2 + c^2}{2} + 2\sqrt{3}A}}$$

$\square$

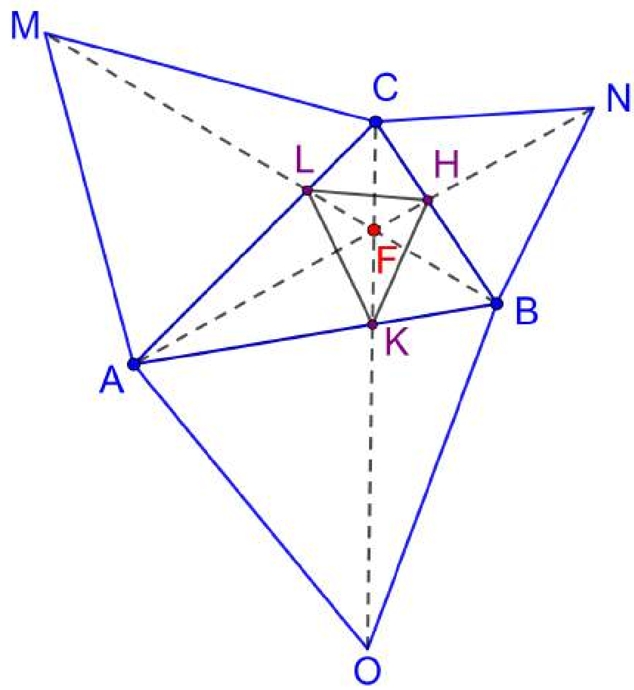
Now we can calculate the area with Heron's formula

$$A = \sqrt{\frac{p}{2} \cdot \left(\frac{p}{2} - a\right)\left(\frac{p}{2} - b\right)\left(\frac{p}{2} - c\right)}$$

so the sum of the distances in terms only of the side lengths of the triangle, letting p be the perimeter of the triangle.

1.5 Wrapping triangles

Let's now consider a triangle ABC and let's draw, thank to the definition, the Fermat point F. We call the triangle LKH **the wrapping triangle of the Fermat point of the triangle** ABC.



We state the following theorem:

Theorem 11. *Given a triangle and is Fermat point F then the Fermat point of the wrapping triangle is again F*

Proof. Due to the Theorem 1 we know that the quadrilateral $AFCM$ in inscribed in a circumference so the angles MFA and MCA are congruent. Therefore the angles MFA measures $60°$. For the same reason $MFC = CFN = NFB = BFO = OFA = 60°$. So the angles $LFK = KFH = LFH = 120°$ and, as stated in Theorem 1, F is the Fermat point of LKH

□

We are able also to calculate the area of the triangle LKH as a function of the lengths of the side of ABC. In fact

Theorem 12. *Given a triangle ABC, with $\overline{AB} = a$, $\overline{BC} = b$, $\overline{AC} = c$ and area A. If $\overline{FA} = x$, $\overline{FB} = y$ and $\overline{FC} = z$ then*

$$A_{LKH} = \frac{\sqrt{3}}{2}\frac{xyz(xy+xz+yz)}{(x+y)(x+z)(y+z)}$$

with

$$x = \frac{a^2+b^2-c^2+\frac{4A}{\sqrt{3}}}{2\ell} \qquad y = \frac{-a^2+b^2+c^2+\frac{4A}{\sqrt{3}}}{2\ell} \qquad z = \frac{a^2-b^2+c^2+\frac{4A}{\sqrt{3}}}{2\ell}$$

and

$$\ell = \sqrt{\frac{a^2+b^2+c^2}{2} + 2A\sqrt{3}}$$

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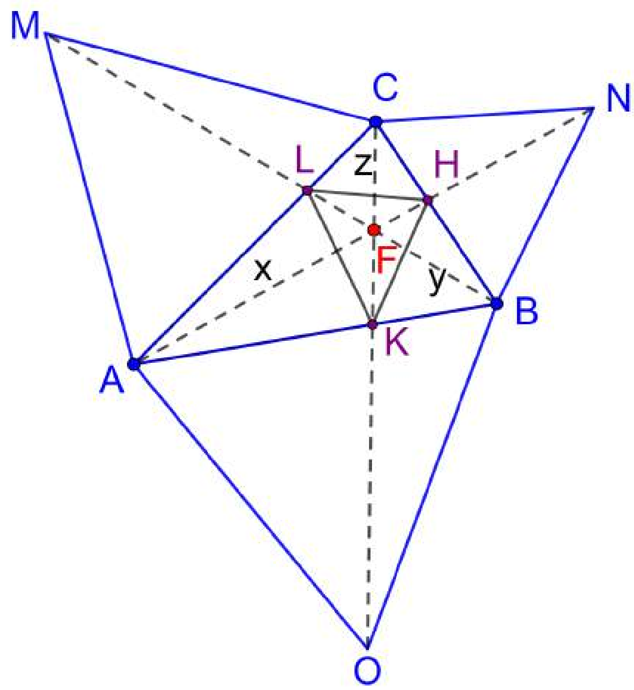


Figure 1.1: Image related to the proof of Theorem 12

Proof. According to Fig. 1.1 we know that $\sin(AFC) = \sin(LFA) = sin(LFC) =$ and

$$\frac{1}{2}xz\sin(AFC) = \frac{1}{2}x\overline{FL}\sin(LFA) + \frac{1}{2}z\overline{FL}\sin(LFC)$$

so

$$xz = x\overline{FL} + z\overline{FL} \qquad \Rightarrow \qquad \overline{FL} = \frac{xz}{x+z}$$

For the same reason

$$\overline{FH} = \frac{yz}{y+z} \qquad \text{and} \qquad \overline{FK} = \frac{xy}{x+y}$$

.

Now

$$\begin{aligned}
A_{LHK} &= A_{LFH} + A_{KFH} + A_{LFK} \\
&= \frac{1}{2}\overline{FL}\cdot\overline{FH}\sin(LFH) + \frac{1}{2}\overline{FK}\cdot\overline{FH}\sin(KFH) + \frac{1}{2}\overline{FL}\cdot\overline{FK}\sin(LFK) \\
&= \frac{\sqrt{3}}{2}\frac{xyz(xy+xz+yz)}{(x+y)(x+z)(y+z)}
\end{aligned}$$

We can observe that, if we call A the area of ABC and $\ell = x+y+z = \overline{MB} = \overline{AN} = \overline{CO}$,

$$\begin{aligned}
A + A_{MAC} &= A_{MFC} + A_{CFB} + A_{FMA} + A_{AFB} \\
&= \frac{1}{2}(\ell - y)\cdot z \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}zy\frac{\sqrt{3}}{2} + \frac{1}{2}x(\ell - y)\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}xy\frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{3}}{4}(x+z)\ell
\end{aligned}$$

and, due to $A_{MAC} = \frac{\sqrt{3}}{4}c^2$,

$$4A + \sqrt{3}c^2 = \sqrt{3}(x+z)\ell$$

.

Similarly

$$\begin{cases}
4A + \sqrt{3}c^2 = \sqrt{3}(x+z)\ell \\
4A + \sqrt{3}b^2 = \sqrt{3}(y+z)\ell \\
4A + \sqrt{3}a^2 = \sqrt{3}(x+y)\ell
\end{cases}$$

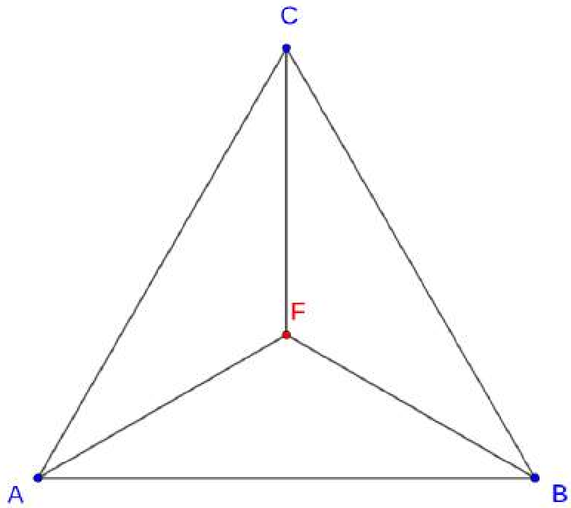
by solving

$$x = \frac{a^2+b^2-c^2+\frac{4A}{\sqrt{3}}}{2\ell} \qquad y = \frac{-a^2+b^2+c^2+\frac{4A}{\sqrt{3}}}{2\ell} \qquad z = \frac{a^2-b^2+c^2+\frac{4A}{\sqrt{3}}}{2\ell}$$

□

1.6 Equilateral triangles

Next we focused on the properties of the Fermat point of an equilateral triangle. Let's consider a generic equilateral triangle ABC with Fermat point F.



We can prove the following theorem about other properties of the Fermat point.

Theorem 13. *The Fermat point F of a equilateral triangle ABC coincides with the circumcenter, the geometric center and the incenter of ABC:*

$$\boldsymbol{F \equiv G \equiv I \equiv C'}$$

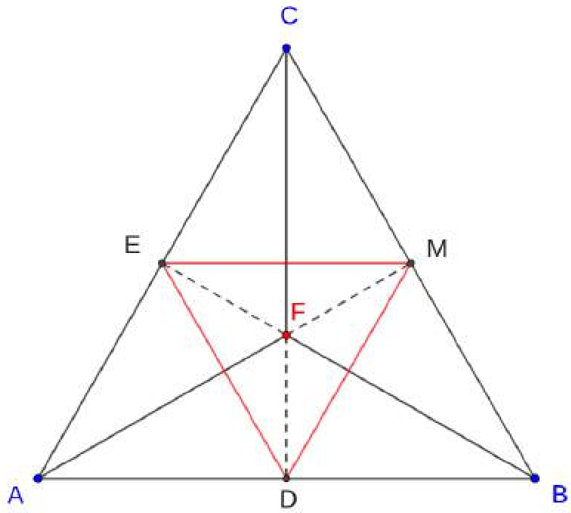
Proof. Let's consider the bisectors of the three angles of the triangle. Since ABC is equilateral, these all intersect on the incenter I, which is the same of the geometric center G and the circumcenter C'.
Let's consider the triangle IBC. Since the inner angles of ABC all measure $\frac{\pi}{3}$, the angles IBC and ICB respectively measure $\frac{\pi}{6}$ and therefore the angle BIC is:

$$IBC = \pi - 2 \cdot \frac{\pi}{6} = \frac{2\pi}{3}$$

We can repeat this process for the other triangles and find that AIC and AIB are also angles of $\frac{2\pi}{3}$. Therefore I is the Fermat point F of ABC.

□

We can then build the *wrapping triangle* DEM of the Fermat point F of ABC.



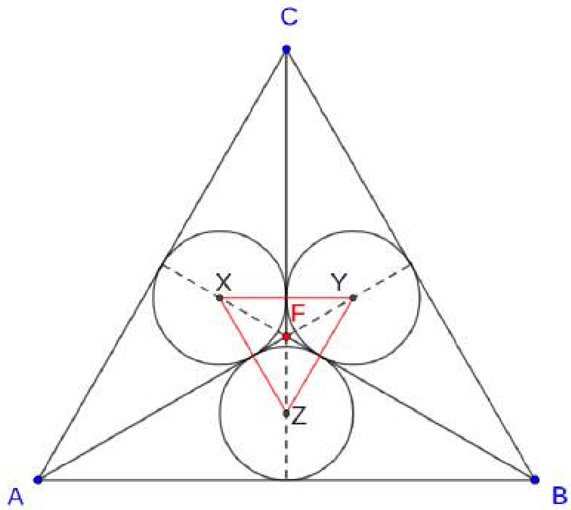
Theorem 14. *The wrapping triangle DEM of the Fermat point of the equilateral triangle ABC is an equilateral triangle.*

Proof. We can prove that EMD is equilateral by simply applying Thales' theorem. All the three sides of the triangle EMD are indeed just half of the sides of ABC and therefore EMD is equilateral.

□

Next, we can build the circumferences inscribed in the triangles AFC, BFC and AFB. We thus obtain

Theorem 15. *Given a equilater triangle ABC, the circumferences inscribed in the triangles AFC, BFC and AFB,having respectively centers in the points X, Y, Z, then*



the triangle XYZ is yet another equilateral triangle, which has the same Fermat point of the ABC:

$$\boldsymbol{XY \cong YZ \cong ZX}$$

$$\boldsymbol{F \equiv F_{XYZ}}$$

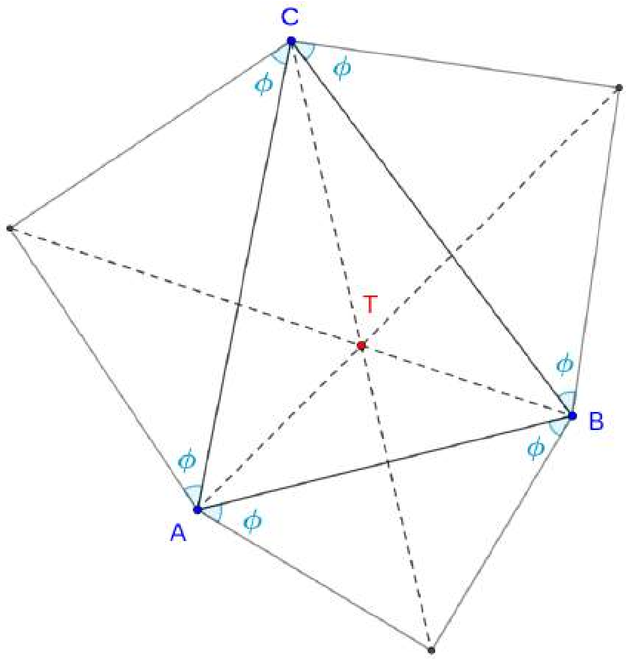
Proof. Let's consider the lines ZC, YA and XB. We can easily prove that they all are bisectors of the angles ACB, CAB and CBA respectively. Therefore, the points X, Y and Z all belong to the lines used to find F and, as in the previous case we have that the Fermat point of the triangle XYZ coincides with F.
To prove that the triangle XYZ is equilateral, we can simply note that the triangles ABF, BFC and CFA are congruent. Given that, we have that the radius of the inscribed circumferences are also congruent and so are the sides of XYZ, being just two times the radius itself.

□

1.7 Generalizing Fermat point: the T points

Recalling the definition of Fermat point (see Definition 1) we can generalize its definition by constructing isosceles triangles on the sides of the triangle in the following way:

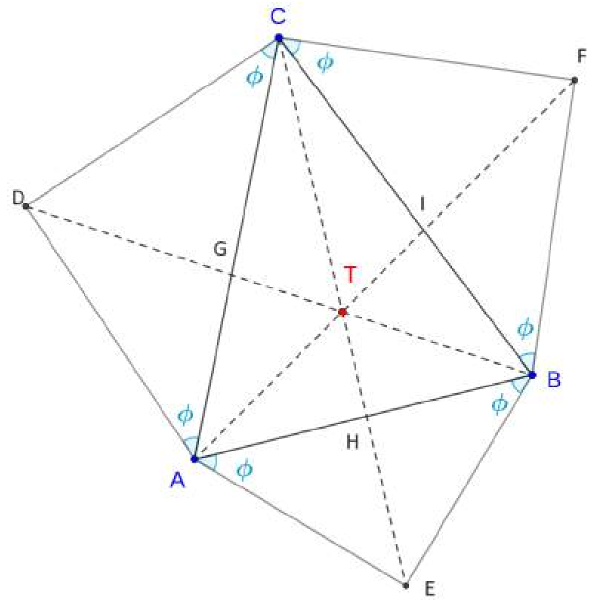
Theorem 16. *In a triangle ABC the segments whose endpoints are a vertex of ABC and the vertex that doesn't belong to the triangle of the isosceles triangle built on the opposite side with an angle $0 < \phi < \frac{\pi}{2}$, all intersect on a single point T.*



Proof. The proof uses Ceva's theorem which states that given a triangle ABC, let the lines , AT, BT, CT be drawn from the vertices to a common point T (not on one of the sides of ABC), to meet opposite sides at I, G, H respectively. Then

$$\frac{CG}{GA} \cdot \frac{AH}{HB} \cdot \frac{BI}{IC} = 1$$

The converse of Ceva's theorem is also true and we are going to use this version.



According to the sine law

$$\frac{CG}{\sin(GDC)} = \frac{DG}{\sin\phi} \qquad \Rightarrow \qquad CG = \frac{DG\sin(GDC)}{\sin\phi}$$

$$\frac{GA}{\sin(ADG)} = \frac{DG}{\sin\phi} \qquad \Rightarrow \qquad GA = \frac{DG\sin(ADG)}{\sin\phi}$$

so

$$\frac{CG}{GA} = \frac{\sin(GDC)}{\sin(ADG)}$$

similarly

$$\frac{AH}{HB} = \frac{\sin(AEH)}{\sin(HEB)} \qquad \frac{BI}{IC} = \frac{\sin(IFB)}{\sin(IFC)}$$

then

$$\frac{CG}{GA}\cdot\frac{AH}{HB}\cdot\frac{BI}{IC} = \frac{\sin(GDC)}{\sin(ADG)}\cdot\frac{\sin(AEH)}{\sin(BEH)}\cdot\frac{\sin(IFB)}{\sin(IFC)}$$

now

$$\frac{DB}{\sin(DCB)} = \frac{CB}{\sin(BDC)} \qquad \Rightarrow \qquad \sin(BDC) = \frac{CB}{DB}\sin(DCB)$$

$$\frac{AB}{\sin(ADB)} = \frac{DB}{\sin(DAB)} \qquad \Rightarrow \qquad \sin(ADB) = \frac{AB}{DB}\sin(DAB)$$

then

$$\frac{\sin(GDC)}{\sin(ADG)} = \frac{CB\sin(DCB)}{AB\sin(DAB)}$$

in the same way

$$\frac{\sin(AEH)}{\sin(BEH)} = \frac{AC\sin(CAE)}{CB\sin(CBE)}$$

and

$$\frac{\sin(BFI)}{\sin(CFI)} = \frac{AB\sin(ABF)}{AC\sin(ACF)}$$

so

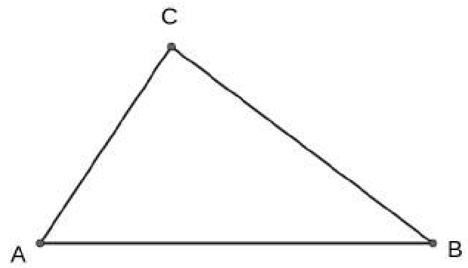
$$\frac{CB\sin(DCB)}{AB\sin(DAB)} \cdot \frac{AC\sin(CAE)}{CB\sin(CBE)} \cdot \frac{AB\sin(ABF)}{AC\sin(ACF)} = 1$$

□

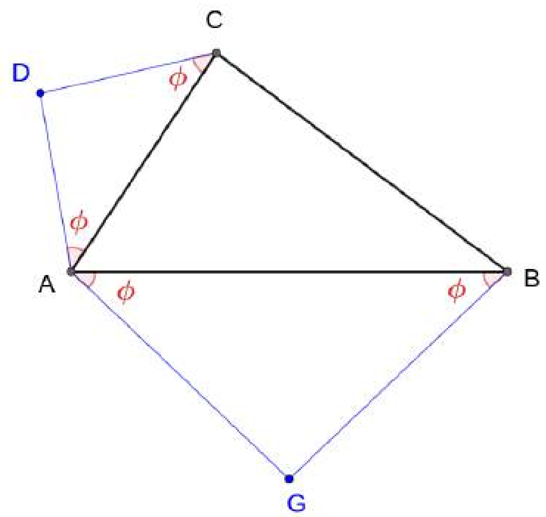
We will call these points the **T points**.

1.7.1 T points analytical coordinates

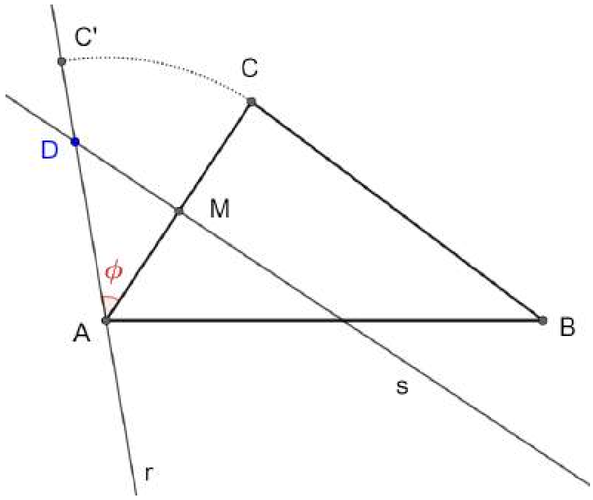
Let's consider a triangle with $A(0;0), B(x_B;0), C(x_C;y_C) \in \mathbb{R}^2$.



Let's now build the isosceles triangles ABG and ACD on the sides AB and AC with an angle $0 < \phi < \frac{\pi}{2}$.



To find the coordinates of $D(x_D;y_D)$, we need to find the intersection between the axis s of the segment AC and the straight line r passing through A and C' (C rotated by ϕ around $A(0;0)$).



Now to find the equation of the straight line r we need the Cartesian coordinates of C'. The rotation matrix R_ϕ to rotate a point $C(x_c; y_c)$ around the origin $A(0;0)$ by an angle ϕ counterclockwise is given by:

$$R_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

By applying the rotation matrix to the point $C(x_c; y_c)$, we obtain the new coordinates $(x'_c; y'_c)$:

$$\begin{bmatrix} x'_c \\ y'_c \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_c \\ y_c \end{bmatrix}$$

Expanding the matrix product, we obtain:

$$\begin{bmatrix} x'_c \\ y'_c \end{bmatrix} = \begin{bmatrix} x_c\cos\phi - y_c\sin\phi \\ x_c\sin\phi + y_c\cos\phi \end{bmatrix}$$

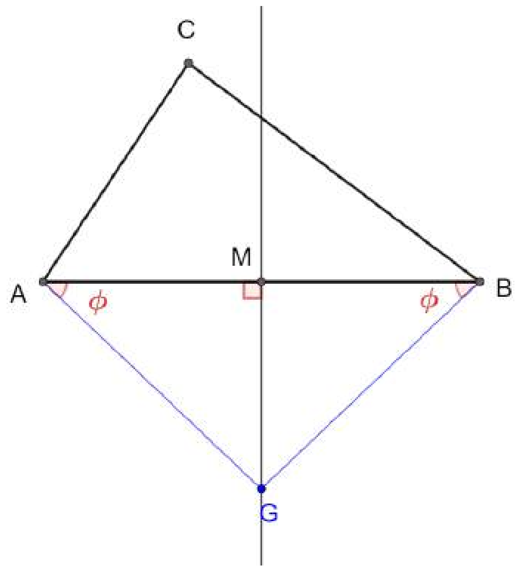
The coordinates of the D points will be the solution of the system that contains the equations of the lines r and s.

$$\begin{cases} y = -\dfrac{x_C}{y_C}x + \dfrac{x_C^2 + y_C^2}{2y_C} \\ \\ y = \dfrac{y_{C'}}{x_{C'}}x \end{cases}$$

$$\boldsymbol{D\left(\frac{x_{C'}(x_C^2 + y_C^2)}{2(x_C x_{C'} + y_C y_{C'})}, \frac{y_{C'}(x_C^2 + y_C^2)}{2(x_C x_{C'} + y_C y_{C'})}\right)}$$

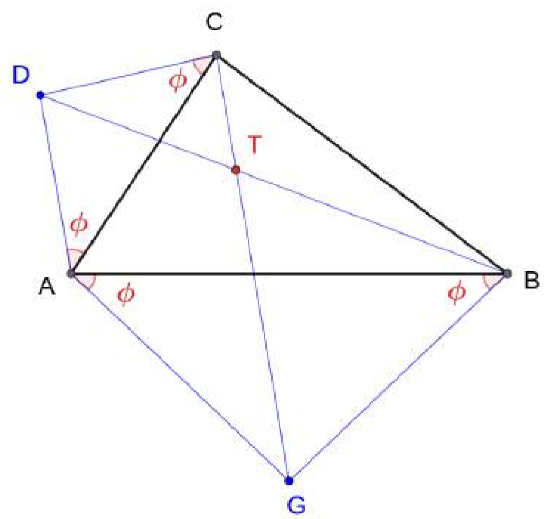
Now we search the coordinates of G.

To find the coordinates of $G(x_G; y_G)$, we can note that the abscissa is the same as the one of the midpoint of the segment AB and the ordinate can be obtained using simple trigonometry on the triangle AMG.



$$\boldsymbol{G\left(\frac{x_B}{2}, -\frac{\tan\phi x_B}{2}\right)}$$

Once we have found the coordinates of D and G, we can continue tracing the line passing through D and B and the line through G and C. Let's call their intersection T.



Finally, by substituting these pairs of points we find the equations of the lines DB and GC which, when put in a system together, will allow us to express the analytical coordinates of $T(x_T; y_T)$.

$$\begin{cases} y = -\dfrac{y_D}{x_D - x_B}x - \dfrac{y_D x_B}{x_D - x_B} \\ \\ y = \dfrac{y_G - y_C}{x_G - x_C}x + \dfrac{x_G y_C - x_C y_G}{x_G - x_C} \end{cases}$$

and finally:

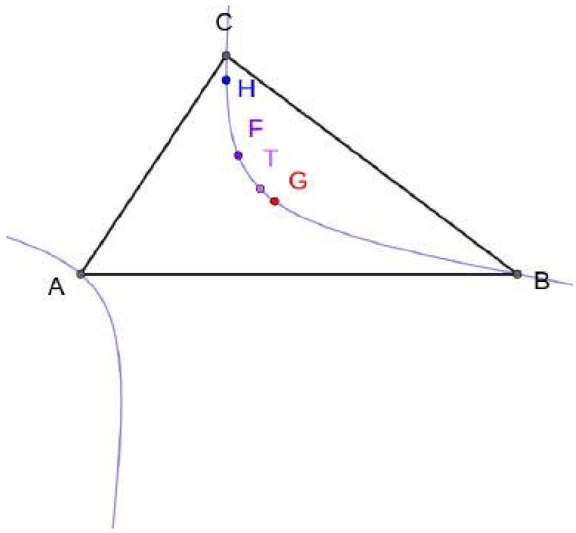
$$\boldsymbol{T}\left(\frac{\boldsymbol{x_B[t(y_C + x_C \cdot \tan\phi) + v]}}{\boldsymbol{t(2y_C + x_B \cdot \tan\phi) + v}}, \frac{\boldsymbol{x_B \cdot v[\tan\phi(x_C - x_B) - y_C]}}{\boldsymbol{(x_B - 2x_C)[t(2y_C + x_B \cdot \tan\phi) + v]}}\right)$$

with

- $t = (x_C - 2x_B)\cos\phi - y_C \cdot \sin\phi$
- $v = (x_C \cdot \sin\phi + y_C \cdot \cos\phi)(x_B - 2x_C)$

1.7.2 T points Hyperbola

Let's consider a generic triangle ABC, by changing the value of ϕ we can notice a geometric locus of T points, which is a hyperbola $\boldsymbol{\mathcal{I}}$. Some of the most famous points such as the Fermat point F, the geometric center G and the orthocentre H belong to it. In fact

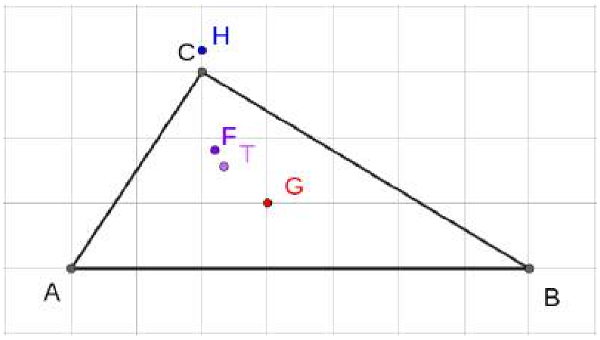


$$G = \lim_{\phi \to 0} T(\phi) \qquad F = T\left(\frac{\pi}{3}\right) \qquad H = \lim_{\phi \to \frac{\pi}{2}} T(\phi) \qquad B = T(-ABC) \qquad C = T(\pi - ACB)$$

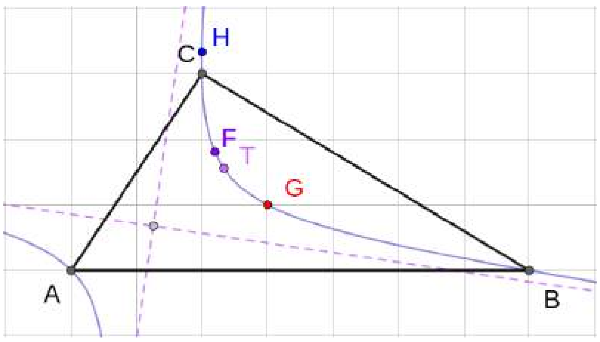
Example

Let's consider again the triangle ABC with $A(0;0)$, $B(7;0)$ and $C(2;3)$. We can calculate the coordinates of the points

$$H\left(2; \frac{10}{3}\right) \qquad G\,(3;1) \qquad T_{\frac{\pi}{4}}\left(\frac{7}{3}; \frac{14}{9}\right) \qquad F\left(\frac{273 + 119\sqrt{3}}{218}; \frac{441 + 427\sqrt{3}}{654}\right)$$



and then we obtain the equation of the hyperbola with equation



$\boldsymbol{\mathcal{I}}: x^2 - y^2 + \dfrac{20}{3}xy - 7x - 7y = 0$
with eccentricity $e = \sqrt{2}$ and center $\left(\dfrac{273}{218}; \dfrac{147}{218}\right)$ called Kiepert hyperbola.

Chapter 2

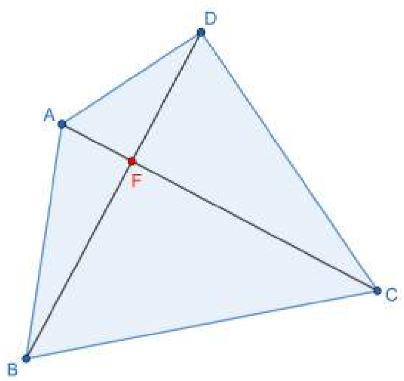
Quadrilaterals

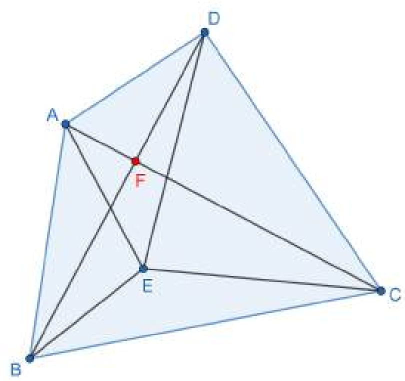
In this chapter we analyse Femat point in quadrilaterals. Since quadrilaterals can be either convex or concave, we analysed both cases. The first thing we did was extending the definition of Fermat point to n sided polygons:

Definition 2. *The Fermat point F of a polygon with n vertices is the point of the polygon which minimizes the sum of the distances from the vertices.*

2.1 Fermat point in convex quadrilaterals

Theorem 17. *In a convex quadrilateral the Fermat point is the intersection of the diagonals.*





Proof. Let's consider a point E inside the quadrilateral $ABCD$. For the triangular inequalities, the diagonal AC is never bigger than $AE + CE$. Similary the diagonal BD is never bigger than $BE + DE$. The sum of the diagonals $AC + BD$ is never bigger than the sum of the segments $AE + BE + CE + DE$.

$$\begin{cases} AC \leqslant AE + CE \\ BD \leqslant BE + DE \\ AC = AF + CF \\ BD = BF + DF \end{cases}$$

$$Substituting:$$

$$\begin{cases} AF + CF \leqslant AE + CE \\ BF + DF \leqslant BE + DE \end{cases}$$

$$Reducing:$$

$$AF + BF + CF + DF \leqslant AE + CE + BE + DE$$

□

2.2 Fermat point in concave quadrilaterals

Theorem 18. *In a concave quadrilateral the Fermat point lies on the vertex of the concavity.*

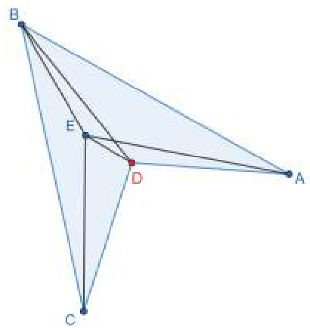


Figure 2.1: The quadrilateral $ABCD$ with point E

Proof. Let's consider the sum of the segments ED and EB. Its minimum occurs when the point E belongs to the segment BD. Now let's consider the sum of the segments EA and EC. Considering that the Fermat point must be inside the quadrilateral, the shortest way to go from A to C is to pass through D. Thus we know that the point E must belong to one of those segments. To satisfy both conditions, the point must coincide with D.

This is summarised by the following inequalities:

$$\begin{cases} AD \leqslant AE + DE \\ CD \leqslant CE + DE \\ AD \leqslant AE \\ CD \leqslant CE \end{cases}$$

□

2.3 Theorems regarding Fermat point in parallelograms

In the next theorems we describe two properties of the Fermat point in parallelograms:

Theorem 19. *In a parallelogram ABCD, the quadrilateral whose vertices are the Fermat points of the triangles AFB, ADF, DCF and BFC is a parallelogram and has the same Fermat point of ABCD.*

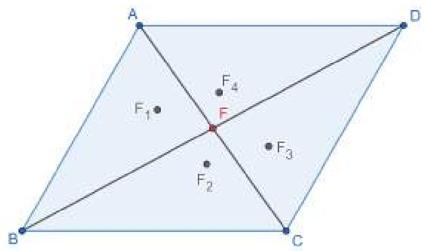


Figure 2.2: A parallelogram ABCD and the parallelogram $F_1F_2F_3F_4$.

Proof. ADF is symmetrical to BCF with respect to F and, similarly, ABF is symmetrical to CDF with respect to F. This is because A is symmetrical to C, B is symmetrical to D and F is symmetrical to itself.

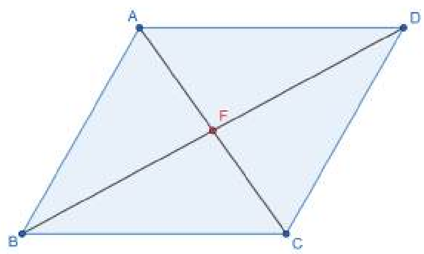


Figure 2.3: The parallelogram ABCD and its Fermat point.

So the Fermat points of ABF (F_1) and CDF (F_3) are symmetrical. So F is the midpoint of the segment F_1F_3.
Similarly, the Fermat points of BCF (F_2) and ADF (F_4) are symmetrical. So F is the midpoint of the segment F_2F_4.

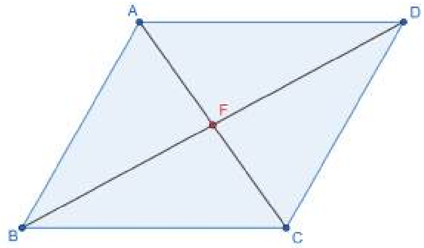


Figure 2.4: The parallelogram ABCD with the Fermat point of ABF, BCF, CDF, ADF

Since F is the midpoint of the digonals of the quadrilateral $F_1F_2F_3F_4$, the quadrilateral $F_1F_2F_3F_4$ is a parallelogram □

Theorem 20. *In a parallelogram ABCD, the Fermat's point of the quadrilateral constructed to the Fermat point of the triangles ABC, ACD, ABD and BCD is a parallelogram and has the same Fermat's point of ABCD.*

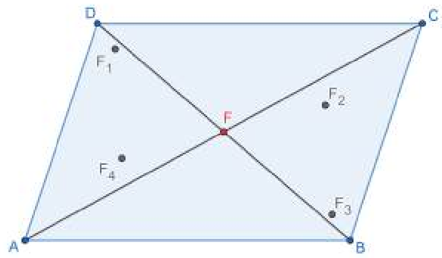


Figure 2.5: The parallelogram ABCD and the parallelogram $F_1F_2F_3F_4$.

Proof. ABC is symmetrical to ACD with respect to F and, similarly, BCD is symmetrical to ABD with respect to F because A is symmetrical to C and B is symmetrical to D.

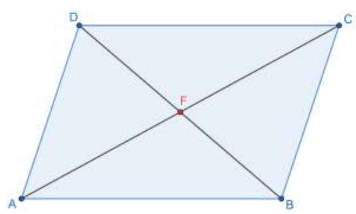


Figure 2.6: The parallelogram ABCD and its Fermat point.

So the Fermat points of ABC (F_3) and ACD (F_1) are symmetrical with F being their center of symmetry and so are the Fermat points of BCD (F_2) and ABD (F_4). So F is the midpoint of the segment F_1F_3 and F_2F_4.

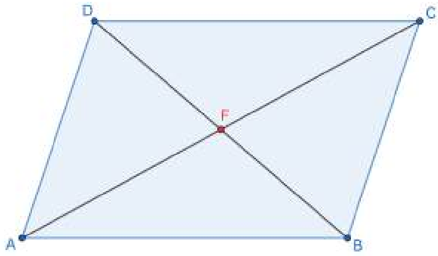


Figure 2.7: The parallelogram ABCD with the Fermat points of ABC, BCD, ACD, ABD

Since F is the midpoint of the diagonals of the quadrilateral $F_1F_2F_3F_4$, the quadrilateral $F_1F_2F_3F_4$ is a parallelogram

$\square$

Chapter 3

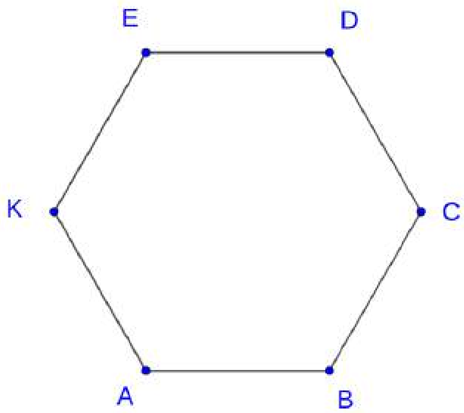
Regular polygons

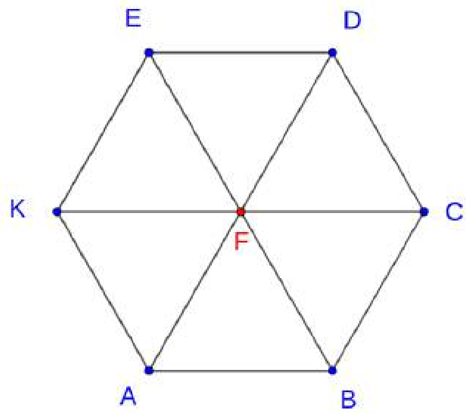
Moving on to regular polygons, we assumed the Fermat point to be the intersection of the diagonals, as it was in the quadrilaterals.

3.1 Fermat point

3.1.1 Even number of sides

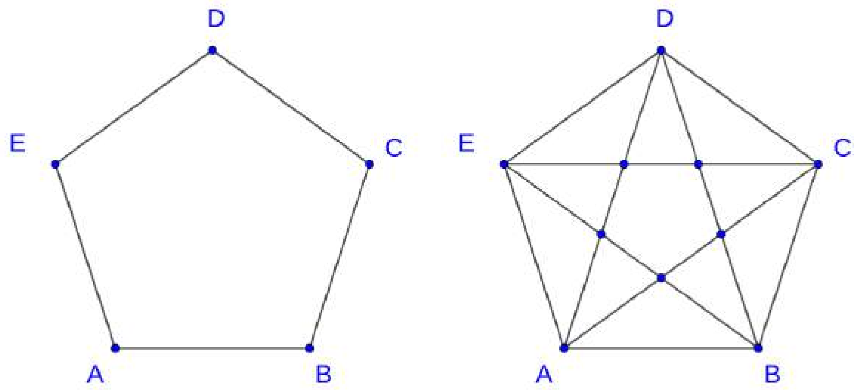
If the polygon has an even number of sides, we can see that the diagonals do intersect in a single point, that being the center of the polygon and its Fermat point.





3.1.2 Odd number of sides

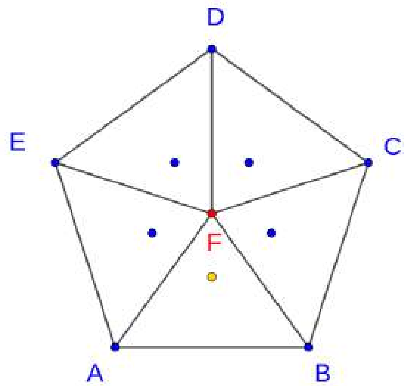
Contrarily, if the polygon has an odd number of sides, the diagonals do not intersect in a single point. Despite this, we were able to prove that indeed, for any regular polygon, the Fermat point coincides with the center of the polygon.



3.1.3 Proof

The proof of both cases proceeds by contradiction:

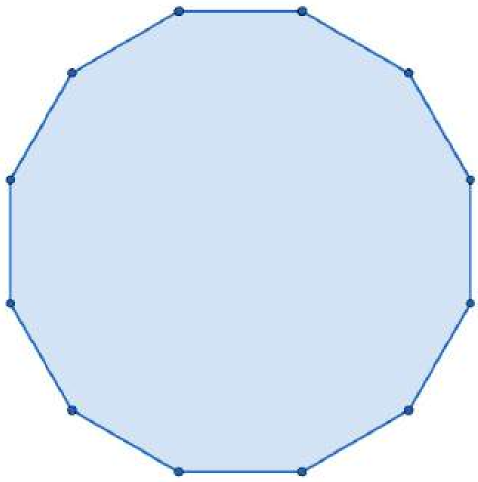
Proof. We first assume the Fermat point is not the center of the polygon and we thus choose a random point inside the polygon.



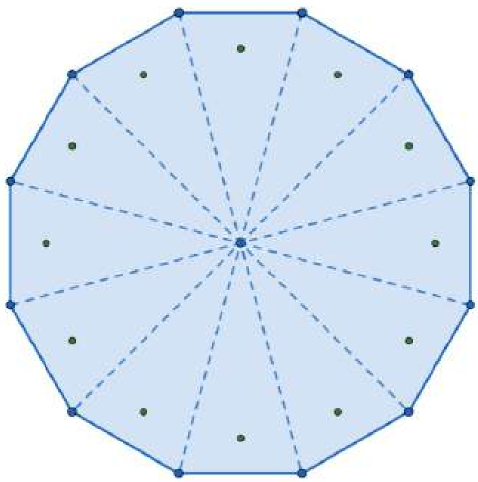
As we can see in the picture, because of the symmetry of the polygon, whatever point we choose is equivalent to the highlighted points.
Since we know the Fermat point is unique (the sum of the distances from the vertices strictly convex and defined in a compact set; see Chapter 4) and that the center of the polygon is the only point that is not equivalent to other symmetric ones, we conclude that, indeed, F coincides with the center of the polygon. □

3.2 Rotated polygons

Let's now consider another regular polygon, with more than three sides, where ℓ_B denotes the length of its sides.



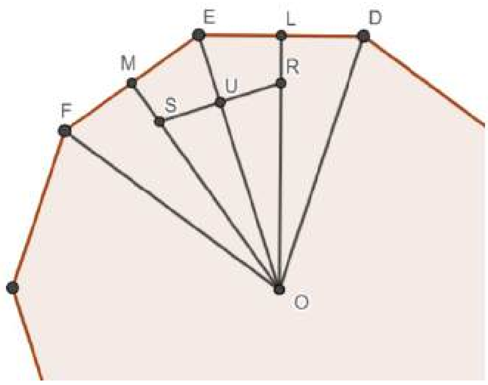
We construct the triangles by connecting the vertices to the center of the polygon, as shown in the picture, and we find their Fermat points.



If we connect them as shown here, we obtain another regular polygon that is similar to the starting one. Furthermore, we can show that:

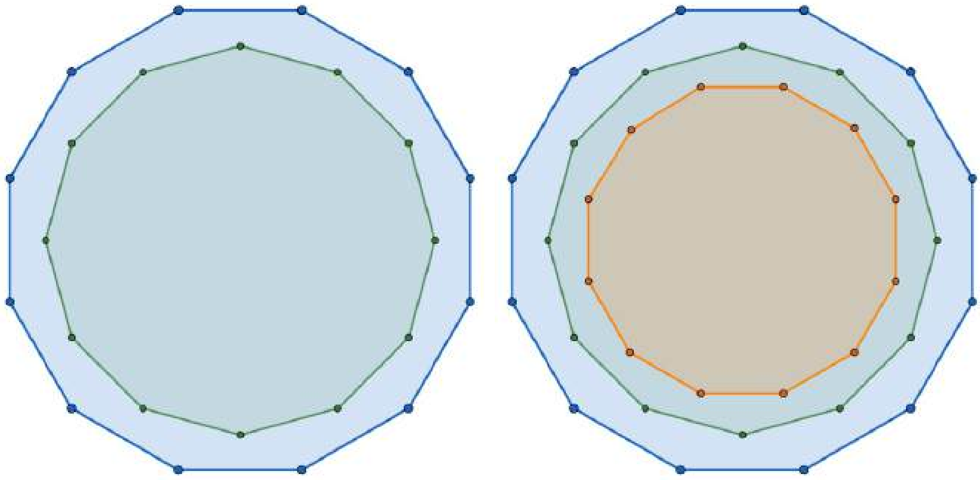
$$\boldsymbol{\ell_S = \ell_B \cdot \frac{2}{\sqrt{3}} \sin\left(\frac{\pi}{3} - \frac{\pi}{n}\right)}$$

where ℓ_S denotes the length of the sides of this polygon. In fact, according to the law of sines in the triangle ERO,



$$\frac{RS}{ED} = \frac{RU}{EL} = \frac{RO}{EO} = \frac{\sin\left(\dfrac{\pi}{3} - \dfrac{\pi}{n}\right)}{\sin\left(\dfrac{2\pi}{3}\right)}$$

Obviously, we can repeat this process iteratively to obtain progressively smaller polygons:



It is worth noting that if we let the number of sides go to infinity, $\dfrac{\pi}{n}$ approaches zero and the whole expression simplifies to just ℓ_B, meaning that the polygon obtained is almost identical to the original one and the Fermat point approaches the base of the triangle according with the theorem 9.

$$\lim_{n \to +\infty} \ell_B \cdot \frac{2}{\sqrt{3}} \sin\left(\frac{\pi}{3} - \frac{\pi}{n}\right) = \ell_B$$

Chapter 4

Approximations

In this last chapter we are going to find an iterative method to approximate the Fermat point of a generic polygon.

4.1 Sum function

Let's consider a polygon with $n \in \mathbb{N} - \{0, 1, 2\}$ vertices $V_1(x_1; y_1), V_2(x_2; y_2)...V_n(x_n; y_n)$.
We define the *Sum function* $S : C \subseteq \mathbb{R}^2 \to \mathbb{R}$ to be the sum of all the euclidean distances of a point in the Cartesian plane from the vertices:

$$\boldsymbol{S(x, y) = \sum_{k=1}^{n} \sqrt{(x - x_k)^2 + (y - y_k)^2}}$$

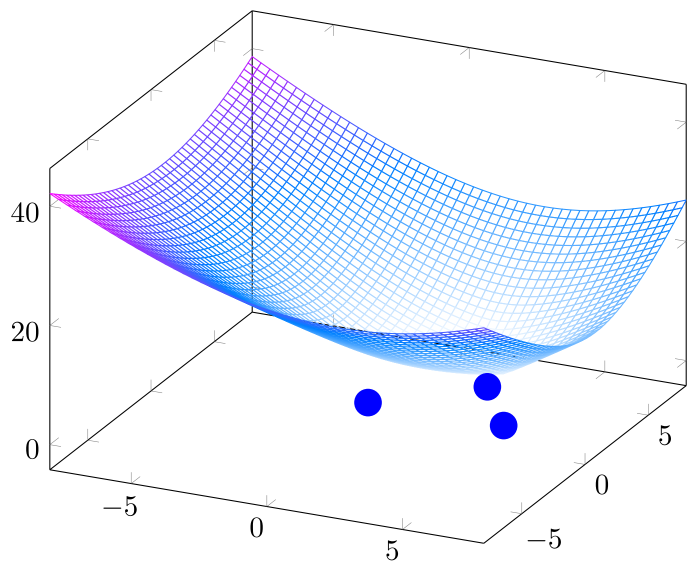


Figure 4.1: Graph of S for a polygon with vertices $V_1(0; 0)$, $V_2(5; 0)$, $V_3(3; 3)$

We can easily see that the function S can be defined in any subset C of $\mathbb{R}^2$:

$$Dom(S) = C \subseteq \mathbb{R}^2$$

4.2 Contour lines

Next we can analyze the behavior of the contour lines $L(S,k)$ of the function S:

$$L(S,k) : \sum_{k=1}^{n} \sqrt{(x-x_k)^2 + (y-y_k)^2} = k$$

We can see that this equation has real solutions only for values of k going from $S(x_F; y_F)$ to $+\infty$:

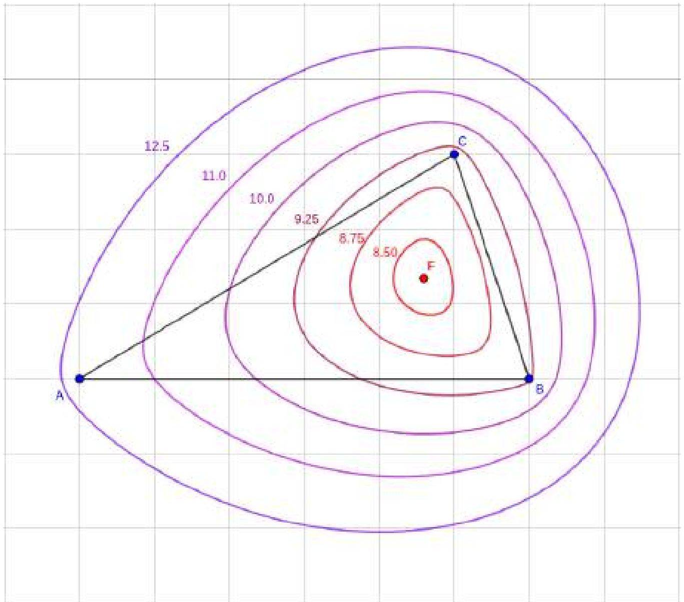


Figure 4.2: Contour lines of the function S

Therefore we can deduce that the image set $Im(S)$ of the function S is:

$$Im(S) = [S(x_F, y_F), +\infty]$$

where $S(x_F, y_F)$ is the value of the function on the Fermat point.

$$\min(S) = S(x_F, y_F)$$

4.3 Gradient descent

In order to find an approximation of the minimum of the function S, we can apply the *gradient descent* method.
We can take the partial derivatives with respect to x and y of $S(x,y)$:

$$\frac{\partial}{\partial x} S(x,y) = \frac{\partial}{\partial x} \sum_{k=1}^{n} \sqrt{(x-x_k)^2 + (y-y_k)^2} = \sum_{k=1}^{n} \frac{x - x_k}{\sqrt{(x-x_k)^2 + (y-y_k)^2}}$$

$$\frac{\partial}{\partial y} S(x,y) = \frac{\partial}{\partial y} \sum_{k=1}^{n} \sqrt{(x-x_k)^2 + (y-y_k)^2} = \sum_{k=1}^{n} \frac{y - y_k}{\sqrt{(x-x_k)^2 + (y-y_k)^2}}$$

and therefore compute the gradient vector as:

$$\boldsymbol{\nabla S(x,y) = \left(\frac{\partial S}{\partial x}; \frac{\partial S}{\partial y}\right) = \left(\sum_{k=1}^{n} \frac{x - x_k}{\sqrt{(x-x_k)^2 + (y-y_k)^2}}; \sum_{k=1}^{n} \frac{y - y_k}{\sqrt{(x-x_k)^2 + (y-y_k)^2}}\right)}$$

We can see that the function S is differentiable on every point of the plane except for the vertices of the polygon.
Since $S(x,y)$ is strictly convex and defined in compact set, it has a unique minimum $F(x_F; y_F)$, whose coordinates can be approximated with the following iterative formula:

$$\boldsymbol{F_{m+1} = F_m - \lambda \nabla S(x_m, y_m)}$$

where $\lambda \in \mathbb{R}^+$.

4.4 Example

Let's now consider the specific case of a triangle ($n = 3$) with vertices $A(0;0)$, $B(3;0)$ and $C(1;1)$. To find a good approximation of the Fermat point F, we can start from a random point, like $F_0(-2;1)$ and compute the next coordinates with the gradient descent method. The results of the first 30 iterations result to be:

$\boldsymbol{m}$	$\boldsymbol{x_{F_m}}$	$\boldsymbol{y_{F_m}}$
0	−2.000	1.000
1	0.875	0.356
2	1.121	0.787
3	0.755	0.702
4	1.512	0.519
5	0.918	0.544
6	1.126	0.706
7	0.929	0.729
8	1.156	0.739
9	0.957	0.715
10	1.068	0.738
11	1.025	0.751
12	1.032	0.762
13	1.031	0.769
14	1.031	0.773

$\boldsymbol{m}$	$\boldsymbol{x_{F_m}}$	$\boldsymbol{y_{F_m}}$
15	1.030	0.775
16	1.030	0.777
17	1.030	0.778
18	1.030	0.779
19	1.030	0.779
20	1.030	0.779
21	1.030	0.779
22	1.030	0.779
23	1.029	0.779
24	1.030	0.779
25	1.029	0.779
26	1.029	0.779
27	1.029	0.779
28	1.029	0.779
29	**1.029**	**0.779**

Since the polygon we have just considered is a triangle with one vertex on the origin, we can compare the coordinates of the approximated point to the values got with the analytical formula we proved before, and compute the errors $\varepsilon_{x_i} = |x_{F_i} - x_F|$ and $\varepsilon_{y_i} = |y_{F_i} - y_F|$ on the x and y coordinates:

m	x_{F_m}	y_{F_m}	ε_{x_i}	ε_{y_i}
0	−2.000	1.000	3.029	0.221
1	0.875	0.356	1.875	0.644
2	1.121	0.787	0.246	0.431
3	0.755	0.702	0.366	0.085
4	1.512	0.519	0.757	0.183
5	0.918	0.544	0.594	0.025
6	1.126	0.706	0.208	0.162
7	0.929	0.729	0.197	0.023
8	1.156	0.739	0.227	0.010
9	0.957	0.715	0.199	0.024
10	1.068	0.738	0.111	0.023
11	1.025	0.751	0.043	0.013
12	1.032	0.762	0.007	0.011
13	1.031	0.769	0.001	0.007
14	1.031	0.773	0.000	0.004

m	x_{F_m}	y_{F_m}	ε_{x_i}	ε_{y_i}
15	1.030	0.775	0.000	0.002
16	1.030	0.777	0.000	0.002
17	1.030	0.778	0.000	0.001
18	1.030	0.779	0.000	0.001
19	1.030	0.779	0.000	0.000
20	1.030	0.779	0.000	0.000
21	1.030	0.779	0.000	0.000
22	1.030	0.779	0.000	0.000
23	1.029	0.779	0.000	0.000
24	1.030	0.779	0.000	0.000
25	1.029	0.779	0.000	0.000
26	1.029	0.779	0.000	0.000
27	1.029	0.779	0.000	0.000
28	1.029	0.779	0.000	0.000
29	**1.029**	**0.779**	**0.000**	**0.000**

We can see that after a few iterations we can get to some approximations of the Fermat point up to more than three digital digits of precision.