HOW MANY POINTS WE CAN PUT ON EARTH, SO THAT THE DISTANCE BETWEEN EACH TWO OF THEM IS AT LEAST 10 000 KILOMETRES?

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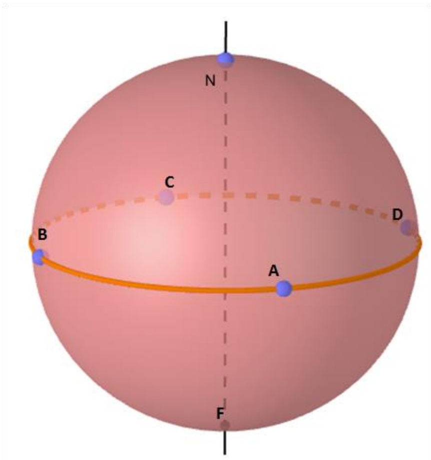


1. Introduction

During the school year 2017-2018 we participated in the Erasmus+ project ***Math&Languages***. We cooperated with students from Souillac. Our task was to solve the title problem, exchange ideas between the Polish and the French team. This paper describes the final results obtained and edited by the Polish part.

The author of the problem is Yohann Genzmer, the researcher of the Souillac team.

2. The problem

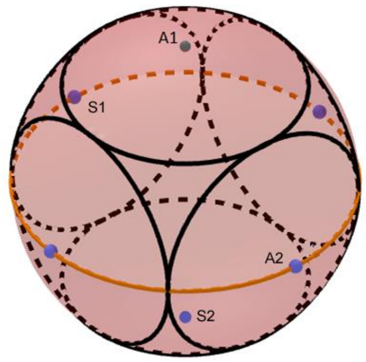
Our problem to solve was to find **<u>the biggest number</u> of points that could be placed on the surface of Earth so that the distance between each two of them is at least 10 000 kilometres.** More precisely, the goal was to solve this problem under the assumptions described below.

We present also our partial results for some ''minimal distances'' different than 10 000 km in the second part of the paper.

3. Assumptions

We assumed that the surface of Earth is a **sphere.** What's more, we rounded down that the **equator measures 40 000** km. But **we fix here <u>the unit</u> to be equal 1000 km**, so it measures 40 (units). ***The distance*** means here **the spherical distance** between two points on Earth, for example the smaller among two lengths of the great circle arcs joining these two points.

4. Notation



We use the following notation, for our purposes:
E - our sphere Earth
c - length of the great circle of *E*
$r = \frac{c}{2\pi}$ - the radius of E
d – the lowest length of the arc between two points
$dist(A, B)$ - the (spherical) distance between points *A, B* on *E*
$Area(S)$- the (spherical) area of a subset *S* of *E*
AB - the length of the interval joining points *A* and *B*
$\widehat{AB}$ - the length of the smaller arc of the great circle joining points *A* and *B* on *E* (note, that $\widehat{AB} = dist(A, B)$ by the definition of the spherical distance...)

For our main problem we assume that:
$\boldsymbol{c = 40}$, $\boldsymbol{d = 10}$.

5. Our result

Our first attempt at finding number of points was trying to place as many points as we could. However, no matter how much we tried, we could not place more than six points. Hence, we first formulated the conjecture that the solution of our problem is **exactly 6**. Then we managed to prove it, so this conjecture became then a theorem!

6. Theorem 1 (*The solution of the main problem*)

The biggest number of points that could be placed on the surface of Earth so that the distance between each two of them is at least 10 000 kilometres equals 6.

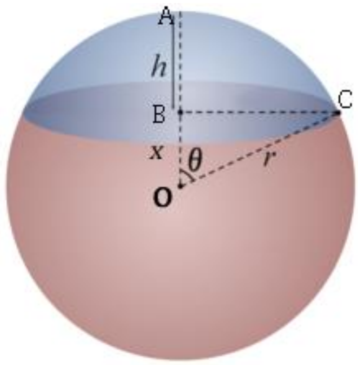
Proof:
To prove that the biggest number of the points is just 6, we had to prove two things: that it is impossible to place more than 6 such points satisfying our conditions, and that it is possible to place 6 points.

a) We cannot place more than 6 points

Suppose that $A_1, A_2, \ldots, A_n$ are some points on the surface of Earth, satisfying the conditions of the problem. Let us create spherical caps $S_1, S_2, \ldots, S_n$ such that each S_i is centred at A_i and has spherical radius $\frac{d}{2}$.
Observe now, that if for some *S1≠S2* there exists a point *X* belonging both to the interior of *S1* and of *S2* (which exactly means that $dist(A_1, X) < \frac{d}{2}$ and $dist(A_2, X) < \frac{d}{2}$), then by the triangle inequality $dist(A_1, A_2) \leq dist(A_1, X) + dist(X, A_2) < \frac{d}{2} + \frac{d}{2} = d$.
But we know, that $dist(A_1, A_2) \geq d$, so such a point X cannot exist!
Such way we have proved, that any two different caps S_1 and S_2 are disjoint or tangential.
We also know, that our caps have the same size, so also their areas are the same. All this allows us to conclude that $Area(E) \geq Area(S_1) + \ldots + Area(S_n) = n \cdot Area(S_1)$.
Hence $n \leq \frac{Area(E)}{Area(S_1)}$.
So the rest of the part **a)** of the proof consists of calculations. We divide the surface of the sphere by the surface of a single spherical cap, to get the upper bound for the number *n* of the points satisfying the rules of the problem.

Calculations:

We use here the well-known formula $Area(S) = 2\pi r h$ for the area of a spherical cap *S* of the sphere *E*, where (see the figure) $h := AB$ is the height of *S*.



Denote also $x := BO$ and recall that in the case of $S = S_1$ we have $\widehat{AC} = \frac{d}{2}$. We also have $r = \frac{40}{2\pi} = \frac{20}{\pi}$, thus $Area(E) = 4\pi r^2 = 4\pi\left(\frac{20}{\pi}\right)^2 = \frac{1600}{\pi}$; $Area(S_1) = 2\pi r h$ and $r = x + h$, so $h = r - x$, $\cos\theta = \frac{x}{r}$, so $h = r(1 - \cos\theta)$.

Moreover $\frac{\theta}{360°} = \frac{\widehat{AC}}{c} = \frac{\widehat{AC}}{40}$, so $\theta = 9\,\widehat{AC}\,°$.

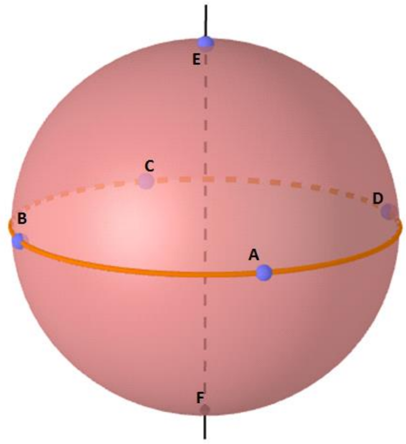
Now we shall use for the first time our assumption that *d = 10,* which gives $\widehat{AC} = 5$, so $\theta = 45°$. Using $\cos 45° = \frac{\sqrt{2}}{2}$ we get

$h = \frac{20}{\pi}(1 - \frac{\sqrt{2}}{2}) = \frac{20 - 10\sqrt{2}}{\pi}$,

and $Area(S_1) = 2\pi\, \frac{20}{\pi}\, \frac{20-10\sqrt{2}}{\pi} = \frac{800-400\sqrt{2}}{\pi}$.

Finally $\frac{Area(E)}{Area(S_1)} = \frac{\frac{1600}{\pi}}{\frac{800-400\sqrt{2}}{\pi}} = 4 + 2\sqrt{2} \leq 6{,}83$ and this means that $n \leq 6{,}83$. **Therefore $\boldsymbol{n} \leq \mathbf{6}$.**

b) We can place at least 6 points



We place points *E* and *F* on the poles of the sphere *E* and we place *A, B, C, D* on the Equator in the same distances $= \frac{40}{4} = 10$. So, the distance between any two of these 6 points equals $\frac{40}{4} = 10$.

Therefore, it is possible to place some 6 points, which finally finishes the proof.

7. The results for other distances

To get some results for the generalization of our problem with the values of *d* different than 10, we first generalized the calculations from part 6 for the upper estimate for the number of points *n*.

Our generalized estimate is based on the considerations from part 6, but it works for every sphere ***E*** with *c= 40* and **for any *d***.

Calculations:

Repeating the argumentation from part 6, we successively get:

$r = \frac{40}{2\pi} = \frac{20}{\pi}$, $Area(E) = 4\pi r^2 = \frac{1600}{\pi}$;

$Area(S_1) = 2\pi r h$, $\quad h = r - x$, $\quad x = r\cos\theta$, $\theta = \frac{9}{2}d$, $h = r(1 - \cos\left(\frac{9}{2}d\right)$,

$Area(S_1) = 2\pi r^2 \left(1 - \cos\left(\frac{9}{2}d\right)\right) = \frac{800(1-\cos(\frac{9}{2}d))}{\pi}$;

$$(*) \quad \boldsymbol{n} \leq \frac{\boldsymbol{Area(E)}}{\boldsymbol{Area(S_1)}} = \frac{\frac{\mathbf{1600}}{\boldsymbol{\pi}}}{\frac{\mathbf{800(1-cos(\frac{9}{2d}))}}{\boldsymbol{\pi}}} = \frac{\mathbf{2}}{\mathbf{1-cos(\frac{9}{2}d)}}$$

Now we shall use the above estimate (*) for several particular values of *d*:

Problem 2: d = **16** units = 16 000 km.

Theorem 2 (*The solution of the Problem 2*)

The biggest number of points that could be placed on the surface of Earth so that the distance between each two of them is at least 16 000 kilometres equals 2.

Proof

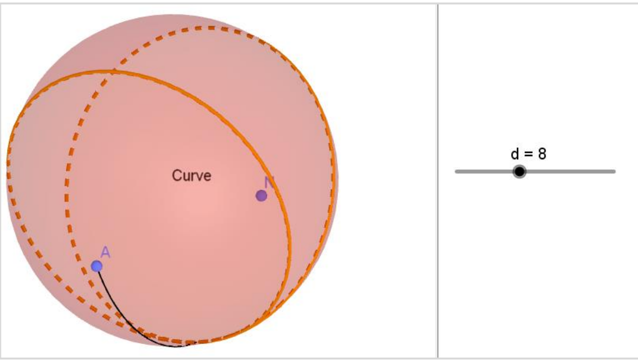
Here we repeat the arguments used in the proof of Theorem 1.

For the part **a)**, by (*) we have (using cos (72°) ≤ 0,31):

$$n \leq \frac{2}{1-\cos(\frac{9}{2}d)} = \frac{2}{1-\cos(72°)} \leq \frac{2}{1-0{,}31} \leq \quad \leq 2.9, \text{ so}$$

$n \leq 2.$

For the part **b)** of the proof it suffices to place two points at the two poles of the sphere *E,* because their distance equals 20, which is greater than 16.



Problem 3: d = **8,3** units = 8 300 km.

Theorem 3 (*The solution of the Problem 3*)

The biggest number of points that could be placed on the surface of Earth so that the distance between each two of them is at least 8 300 kilometres equals 8 or 9.

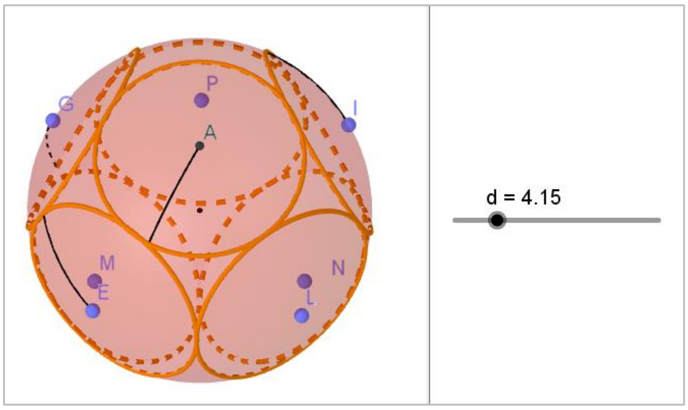
Proof

Here we repeat again the arguments used in the proof of Theorem 1. For the part **a)**, by (*) we have (using cos (37,35°) ≤ 0,795):

$$n \leq \frac{2}{1-\cos\left(\frac{9}{2}d\right)} = \frac{2}{1-\cos(37{,}35°)} \leq$$

$$\leq \frac{2}{1-0{,}795} \leq 9{,}8, \text{ so } n \leq 9.$$

Using Geogebra we can place 8 points (see figure).



Problem 4: d = **7,8** units = 7 800 km.

Theorem 4 (*The solution of the Problem 4*)

The biggest number of points that could be placed on the surface of Earth so that the distance between each two of them is at least 7 800 kilometres equals 9, 10 or 11.

Proof

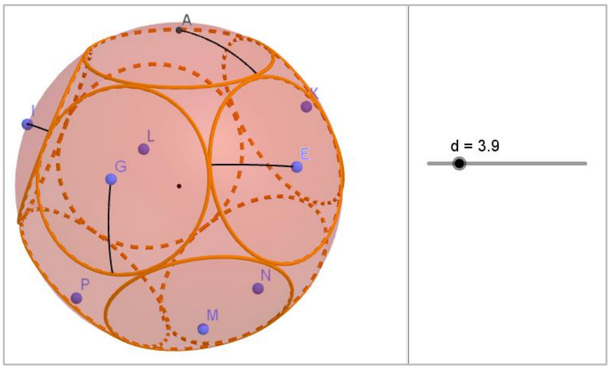
We repeat here way of argumentation from the proof of Theorem 1.

For the part **a)**, by (*) we have (using cos (35,1°) ≤ 0,82):

$$n \leq \frac{2}{1-\cos\left(\frac{9}{2}d\right)} = \frac{2}{1-\cos(35{,}°)} \leq$$

$$\leq \frac{2}{1-0{,}82} \leq 11{,}2, \text{ so } n \leq 11.$$

Using Geogebra we can place 9 points (see figure).



8. Summary

We managed to solve Problem 1 and 2 in details. Our solutions of Problems 3 and 4 are only partial: we do not know the exact number, but we know only some range, consisting of two or three possible solutions. We suppose that the difficulty is as follows. While trying to place more points, we use more space. We suppose that exact solutions are the smallest numbers from the range (those found by Geogebra): 8 for Problem 3 and 9 for Problem 4.