Breeding (like) rabbits

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1. PRESENTATION OF THE RESEARCH TOPIC

In this paper, we approached the “Breeding (like) rabbits” problem, which has always been a popular subject in the mathematics’ world. Thus, we intended to study the evolution of a group of rabbits after a set period of time. To solve the problem, we started with an ideal case, where the mortality rate is null, then we considered a more realistic approach, introducing other variables.

2. BRIEF PRESENTATION OF THE CONJECTURES AND RESULTS OBTAINED

A certain species of rabbits may reach maturity after only one month of life, after which they can breed. However, not all the rabbits will reach maturity within one month. Let’s suppose that, for this species, the grow up rate within one month is p ($0 \leq p \leq 1$). We also assume that the birth rate for this species of rabbits is b, while d and D are the death rates for immature and adult rabbits, respectively.
In the first month, we start with m male-female rabbit pairs. Build a mathematical model for the dynamics of the rabbit population after n months. For some specific values of the above parameters, draw a graph for the evolution of the number of rabbits within the next few years. Find the percentages of adults in this species after a large number of years.

3.THE SOLUTION

Firstly, we decided to start with an easier problem, which we can adjust to ours. Fibonacci's Rabbits is a problem that Fibonacci investigated in the year 1202. It was about how fast rabbits could breed in ideal circumstances. The problem included a newly-born pair of rabbits, one male, one female, that are put on a field. The rabbits are able to mate at the age of one month so that at the end of the second month a female can produce another pair of rabbits. Let us assume that our rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on.

So, we observe that at the end of the first month, they mate, but there is still one only 1 pair. In the next month the female produces a new pair, so now there are 2 pairs of rabbits in the field. At the end of the third month, the original female produces a second pair, making 3 pairs in all in the field. In the fourth month the original female has produced yet another new pair, the female born two months ago produces her first pair also, making 5 pairs.

If we let $f(n)$ mean "the number of pairs of rabbits in the field at the start of the *n*-th month", we will show that $f(1)=1$, $f(2)=1$ and , $f(n)=f(n-1)+f(n-2)$, $n>2$ which is exactly the definition of the Fibonacci sequence (which also has $f(0)=0$). All the rabbits from the previous months survive, so there are at least $f(n-1)$ of them. Any rabbit (pair) that is alive 2 months ago is now able to produce a new pair, and we assume they always will and each will produce only one new pair per month. Thus, the number of newly born pairs is the same as the number of pairs alive 2 months ago: $f(n-2)$. Since all the rabbits were alive last month or are newly born this month, we have: $f(n)=f(n-1)+f(n-2)$, for $n>2$.

We shall start with the Fibonacci sequence:

(*) $$x_{n+1}=x_n+x_{n-1},\ n\geq 1, \text{ where } x_0=0,\ x_1=1.$$

a second order recurrence.

In order to find the solution for $(*)$, meaning that we have to find the expression for x_n, we will start by solving the following equation:

$$\alpha^2-\alpha-1=0,\ \alpha\in\mathbb{R}.$$

This equation has the solutions $\alpha_1=\frac{1-\sqrt{5}}{2}$ and $\alpha_2=\frac{1+\sqrt{5}}{2}$.

If we go back to our equation, we can observe that $\alpha_1+\alpha_2=1$ and $\alpha_1\alpha_2=-1$. So, our recurrence formula can be modified to:

$$x_{n+1}=(\alpha_1+\alpha_2)x_n-\alpha_1\alpha_2 x_{n-1},\ n\geq 1,$$

or

$$x_{n+1}-\alpha_1 x_n=\alpha_2(x_n-\alpha_1 x_{n-1}),\ n\geq 1.$$

We deduce that $(y_n)_{n\geq 1}$, with $y_n=x_n-\alpha_1 x_{n-1}$, $n\geq 1$, is going to check the following equation.

$$y_{n+1}=\alpha_2 y_n,\ n\geq 1,$$

which is a geometric progression with a ratio of α_2.

So,

$$y_n=\alpha_2^{n-1}y_1=\alpha_2^{n-1},\ n\geq 1,$$

(because $y_1=x_1-\alpha_1 x_0=1$).

We will have:

$$\begin{cases} x_n - \alpha_1 x_{n-1} = \alpha_2^{\ n-1} \\ x_{n-1} - \alpha_1 x_{n-2} = \alpha_2^{\ n-2} \Big| \cdot \alpha_1^1 \\ x_{n-2} - \alpha_1 x_{n-3} = \alpha_2^{\ n-3} \Big| \cdot \alpha_1^2 \\ \dots \\ x_2 - \alpha_1 x_1 = \alpha_2^{\ 1} \Big| \cdot \alpha_1^{n-2} \\ x_1 - \alpha_1 x_0 = \alpha_2^{\ 0} \Big| \cdot \alpha_1^{n-1} \end{cases}$$

When we add the equations above, that have been already multiplied with the indicated numbers, we will obtain:

$$x_n = \alpha_2^{n-1} + \alpha_2^{n-2}\alpha_1 + \alpha_2^{n-3}\alpha_1^{\ 2} + \ldots + \alpha_2\alpha_1^{n-2} + \alpha_1^{n-1}$$
$$= \frac{\alpha_2^n - \alpha_1^n}{\alpha_2 - \alpha_1} = \frac{1}{\alpha_2 - \alpha_1}\left(\alpha_2^n - \alpha_1^n\right)$$

Thus, we can see that x_n is:

$$\boxed{x_n = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n, n \geq 1}.$$

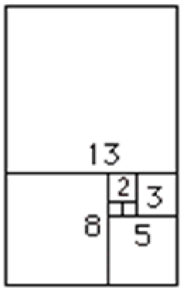
If we annotate $\varphi = \frac{1+\sqrt{5}}{2}$, we can rewrite x_n as:

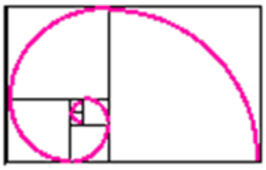
$$x_n = \frac{1}{\sqrt{5}} \cdot \left(\varphi^n - (1-\varphi)^n\right) = \frac{1}{\sqrt{5}} \cdot \left(\varphi^n - \left(-\frac{1}{\varphi}\right)^n\right), \ (\forall) n \geq 1.$$

The number $\varphi = \frac{1+\sqrt{5}}{2}$ is called the *golden ratio*.

Let us show you some information about this famous number.

The value of φ, also called *golden mean*, the *golden number* or the *golden section*, is approximatively 1.618034. It is the ratio of a line segment cut into two pieces of different lengths such that the ratio of a line segment to that of the longer segment is equal to the ratio of the longer segment to the shorter segment. The origin of this number can be traced back to *Euclid*, who mentions it as the *extreme and mean ratio* in his famous book *Elements*. Letting the length of the shorter segment be 1 unit and the length of the longer segment be *x* units, we get the equation: $\frac{x+1}{x} = \frac{x}{1}$, or $x^2 - x - 1 = 0$, for which the positive root is $x = \frac{1+\sqrt{5}}{2}$. The ancient Greeks recognized this *dividing* or *sectioning* property, a phrase that was ultimately shortened to simply *the section*. It was more than 2000 years later that both *ratio* and *section* were designated as *golden* by German mathematician Martin Ohm in 1835. The Greeks also had observed that *golden ratio* provided the most aesthetically pleasing proportion of sides of a rectangle. A *golden rectangle* is a rectangle whose side lengths are in the *golden ratio* ($1:\varphi$). We could also build the aforementioned shape by using the Fibonacci numbers as sizes for the square's units. (1, 1, 2, 3, 5, 8, 13, …). Starting with two squares of size 1 next to each other, one can keep adding squares as long as each new square has a side which is as long as the sum of the last two square's sides.





By also drawing a spiral (with its starting point in the first square) which is made up of a quarter of a circle in each square we get a pretty accurate representation or approximation of a shape often found in nature. The spiral-in-the-squares makes a line from the center of the spiral increased by a factor of the golden number in each square.

So, for $(*)\, x_{n+1} = x_n + x_{n-1}$, where $n \geq 1$, we have that:

$$x_n = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n, \text{ where } n \geq 1.$$

We notate: $c_n = x_{n-1}$ the number of pairs of children at step *n* and $a_n = x_n$ the number of pairs of adults at step *n*.

So, we rewrite $(*)$ as:

$$\begin{pmatrix} a_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_n \\ c_n \end{pmatrix}, \text{ where } n \geq 1.$$

Now, we will consider a more general case (a more realistic one too).

Let *D* be the rate of mortality of the adults, *d* the rate of mortality of children, *p* the rate of reaching maturity and *b* the fertility rate of the adults. We want to find out the number a_n the number of adults at the *n*-th month and c_n the number of children at the *n*-th month, for all $n \geq 1$.

Firstly, we may observe that:

$$\begin{cases} a_{n+1} = (1-D) \cdot a_n + p \cdot c_n \\ c_{n+1} = b \cdot a_n + (1-d-p) \cdot c_n \end{cases}, \text{ for all } n \geq 1.$$

This two equations may be translated in the following matriceal system:

$$\begin{pmatrix} a_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 1-D & p \\ b & 1-d-p \end{pmatrix} \cdot \begin{pmatrix} a_n \\ c_n \end{pmatrix}, \text{ for all } n \geq 1.$$

We will now have $A = \begin{pmatrix} 1-D & p \\ b & 1-d-p \end{pmatrix}$, and $X_n = \begin{pmatrix} a_n \\ c_n \end{pmatrix}$, so

$$X_{n+1} = A \cdot X_n, \text{ for all } n \geq 1.$$

By mathematical induction, we obtain that:

$$X_{n+1} = A^n \cdot X_1, \text{ for all } n \geq 1.$$

That equation shows us that, knowing how to determine A^n, we can easily find out X_n

Now, we will present a mathematical method to find a general formula for A^n. We will now work with eigenvalues. Let λ_1 and λ_2 be the eigenvalues of the matrix *A*, which can easily be determined as the solutions of the equation:

$$x^2-\operatorname{Tr}(A)\cdot x+\det(A)=0, \text{ where}$$
$$\operatorname{Tr}(A)=1-D+1-d-p=2-(d+p+D)$$
$$\det(A)=(1-D)\cdot(1-d-p)-pb$$

We shall prove the following theorem:

$$A^n=\begin{cases}\lambda_1^nB+\lambda_2^nC, \text{ if } \lambda_1\neq\lambda_2\\ \lambda_1^nB+(n-1)\lambda_2^{n-1}C, \text{ if } \lambda_1=\lambda_2\end{cases},$$

where *B* and *C* are two squared matrix that can be determined by cases $n=1$ and $n=2$.

Matrix *A* checks the following relation:

$$A^2-\operatorname{Tr}(A)\cdot A+\det(A)\cdot I_2=O_2,$$

Where $\operatorname{Tr}(A)=a+d$ and $\det(A)=ad-bc$.

If we multiply the relation above with A^{n-1} we will obtain:

$$A^{n+1}-\operatorname{Tr}(A)\cdot A^n+\det(A)\cdot A^{n-1}=O_2.$$

From this, we can conclude that

(1) $$A^{n+1}=\operatorname{Tr}(A)\cdot A^n-\det(A)\cdot A^{n-1}=O_2.$$

Considering $A^n=\begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix}$ and relation (1), we will obtain:

$$\begin{cases}a_{n+1}=\operatorname{Tr}(A)\cdot a_n-\det(A)\cdot a_{n-1}\\ b_{n+1}=\operatorname{Tr}(A)\cdot b_n-\det(A)\cdot b_{n-1}\\ c_{n+1}=\operatorname{Tr}(A)\cdot c_n-\det(A)\cdot c_{n-1}\\ d_{n+1}=\operatorname{Tr}(A)\cdot d_n-\det(A)\cdot d_{n-1}\end{cases},\ n\geq 2.$$

All of them can be translated to a single recurrence relation:

$$x_{n+1}=\operatorname{Tr}(A)\cdot x_n-\det(A)\cdot x_{n-1},\ n\geq 2.$$

The characteristic equation is $\lambda^2-\operatorname{Tr}(A)\cdot\lambda+\det(A)=0$. So, we will have 2 cases:

I. If $\lambda_1\neq\lambda_2$ we will have $x_n=\alpha_x\lambda_1^n+\beta_x\lambda_2^n$, where $\alpha_x,\beta_x\in\mathbb{C}$. So,

$$\begin{cases}a_n=\alpha_a\lambda_1^n+\beta_a\lambda_2^n\\ b_n=\alpha_b\lambda_1^n+\beta_b\lambda_2^n\\ c_n=\alpha_c\lambda_1^n+\beta_c\lambda_2^n\\ d_n=\alpha_d\lambda_1^n+\beta_d\lambda_2^n\end{cases}$$

the matrixes $B,\ C\in M_2(\mathbb{C})$ exist,

$$B=\begin{pmatrix}\alpha_a & \alpha_b\\ \alpha_c & \alpha_d\end{pmatrix},\ C=\begin{pmatrix}\beta_a & \beta_b\\ \beta_c & \beta_d\end{pmatrix}.$$

Thus,

$$A^n=\lambda_1^nB+\lambda_2^nC.$$

II. If $\lambda_1 = \lambda_2$ we will have $x_n = \alpha_x \lambda_1^n + \beta_x n \lambda_1^{n-1}$, where $\alpha_x, \beta_x \in \mathbb{C}$. So,

$$\begin{cases} a_n = \alpha_a \lambda_1^n + \beta_a n \lambda_1^{n-1} \\ b_n = \alpha_b \lambda_1^n + \beta_b n \lambda_1^{n-1} \\ c_n = \alpha_c \lambda_1^n + \beta_c n \lambda_1^{n-1} \\ d_n = \alpha_d \lambda_1^n + \beta_d n \lambda_1^{n-1} \end{cases}$$

the matrixes $B, C \in M_2(\mathbb{C})$ exist,

$$B = \begin{pmatrix} \alpha_a & \alpha_b \\ \alpha_c & \alpha_d \end{pmatrix},\ C = \begin{pmatrix} \beta_a & \beta_b \\ \beta_c & \beta_d \end{pmatrix}.$$

Thus,

$$A^n = \lambda_1^n B + \lambda_1^{n-1} n C .$$

B and C can be easily determined by the cases $n = 1$ and $n = 2$ where A will be $A^0 = I_2$ and $A^1 = A$.

Let's see if we can predict how the population will end up, based on the values of d, p, D, b. Let a_n be the number of adults on day n and c_n be the number of children on day n.

$$\begin{aligned} x_n &= a_n + c_n \\ x_{n+1} &= a_{n+1} + c_{n+1} \\ &= (1-D)a_n + pc_n + ba_n + (1-d-p)c_n \\ &= a_n(1-D+b) + c_n(1-d) \\ &= x_n + a_n(b-D) - dc_n \end{aligned}$$

Therefore, the way the population changes from day n to day $n+1$ is determined by the relation of $a_n(b-D) - dc_n$ with 0. We observe that:

$$a_n(b-D) - dc_n > 0 \Leftrightarrow a_n(b-D) > dc_n \Leftrightarrow \frac{b-D}{d} > \frac{c_n}{a_n}.$$

The formulas above are true for $d \neq 0$ (if $d = 0$, the population will always grow). We notice that the relation of $a_n(b-D) - dc_n$ with 0 is equivalent with the relation of $\frac{b-D}{d}$ with $\frac{c_n}{a_n}$. To find out what happens after a very large number of days, all we have to do is compare $\lim_{n\to\infty} \frac{c_n}{a_n}$ with $\lim_{n\to\infty} \frac{b-D}{d} = \frac{b-D}{d}$, and we have three cases:

$\lim_{n\to\infty}\frac{c_n}{a_n} > \frac{b-D}{d}$: the population will drop to 0;

$\lim_{n\to\infty}\frac{c_n}{a_n} = \frac{b-D}{d}$: the population will stabilize itself to a positive value;

$\lim_{n\to\infty}\frac{c_n}{a_n} < \frac{b-D}{d}$: the population will grow to infinity.

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To better visualize how the formulas above work and how the rabbit population is changing based on the different values we assign the parameters, we created a Python program to help us achieve that.

For the data manipulation, we created a function for multiplying the 2×2 matrices *A* and *B*, which returns the final matrix *C*.

```
def multiply(A,B):
    C=[[0,0],[0,0]]
    for i in range(0,2):
        for j in range(0,2):
            for x in range(0,2):
                C[i][j]=C[i][j]+A[i][x]*B[x][j]
    return C
```

After this, we set up the 4 parameters and the number of months we want to see the plot. In this example, we will assign them according to the Fibonacci case.

```
D=0.0  # adult mortality rate
p=1.0  # child maturiaztion rate
b=1.0  # adult fertility rate
d=0.0  # child mortality rate
n=15   # number of months we want to see on the graphic
```

Using the formula $\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 1-D & p \\ b & 1-d-p \end{pmatrix}^{n-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, to obtain the result for each step, we will assign the variable matrix to *A*, and the base cases matrix to *B*, then we will successively multiply *B* with *A*, using the "multiply" function. The current result will always be in the matrix *B*.

```
A=[[1-D,p],[b,1-d-p]]
B=[[1,0],[0,0]]
x=[] # the total number of rabbits
a=[] # the number of adults
c=[] # the number of children
z=[] # the fraction adults/children
```

```
valp=0; limit=""
for i in range(1000):
    if i<=n:
        x.append(B[0][0]+B[1][0])
        a.append(B[0][0])
        c.append(B[1][0])
        z.append(B[1][0]/B[0][0])

    valr=B[1][0]/B[0][0]
    aux=valp
    valp=B[0][0]+B[1][0]
    B=multiply(A,B)
```

Note: we calculate 1000 steps for better accuracy of the limits, but we only show the results for the first n.

In the end, we used a bunch of functions from the “matplotlib library” for turning these data into plots, and we obtained this for the attributed values:

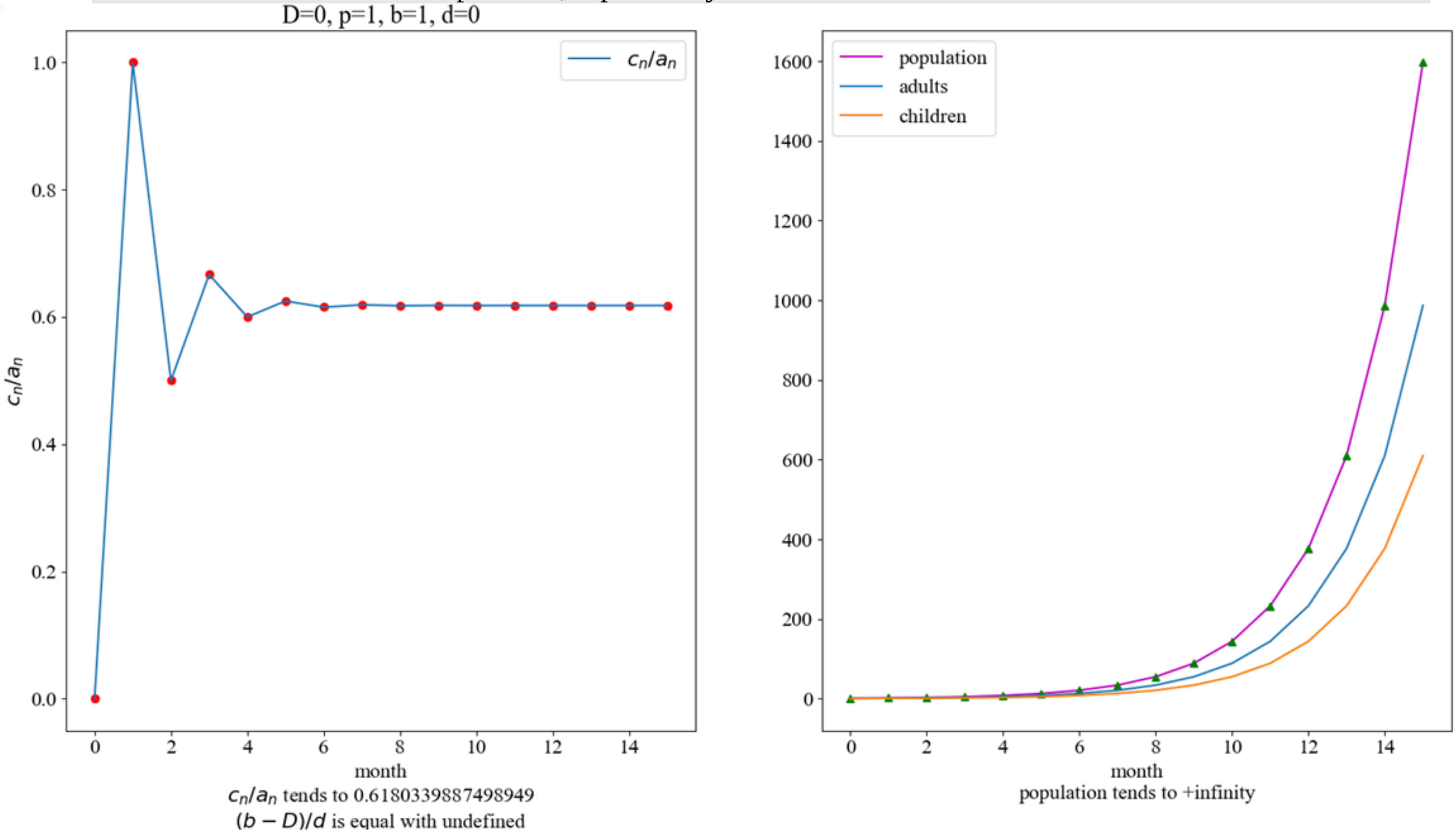

Figure 1. The ratio *children*:*adults* and the dynamics of the rabbit population for the Fibonacci sequence

We can insert any values we want and observe how the graphic changes under different variables.

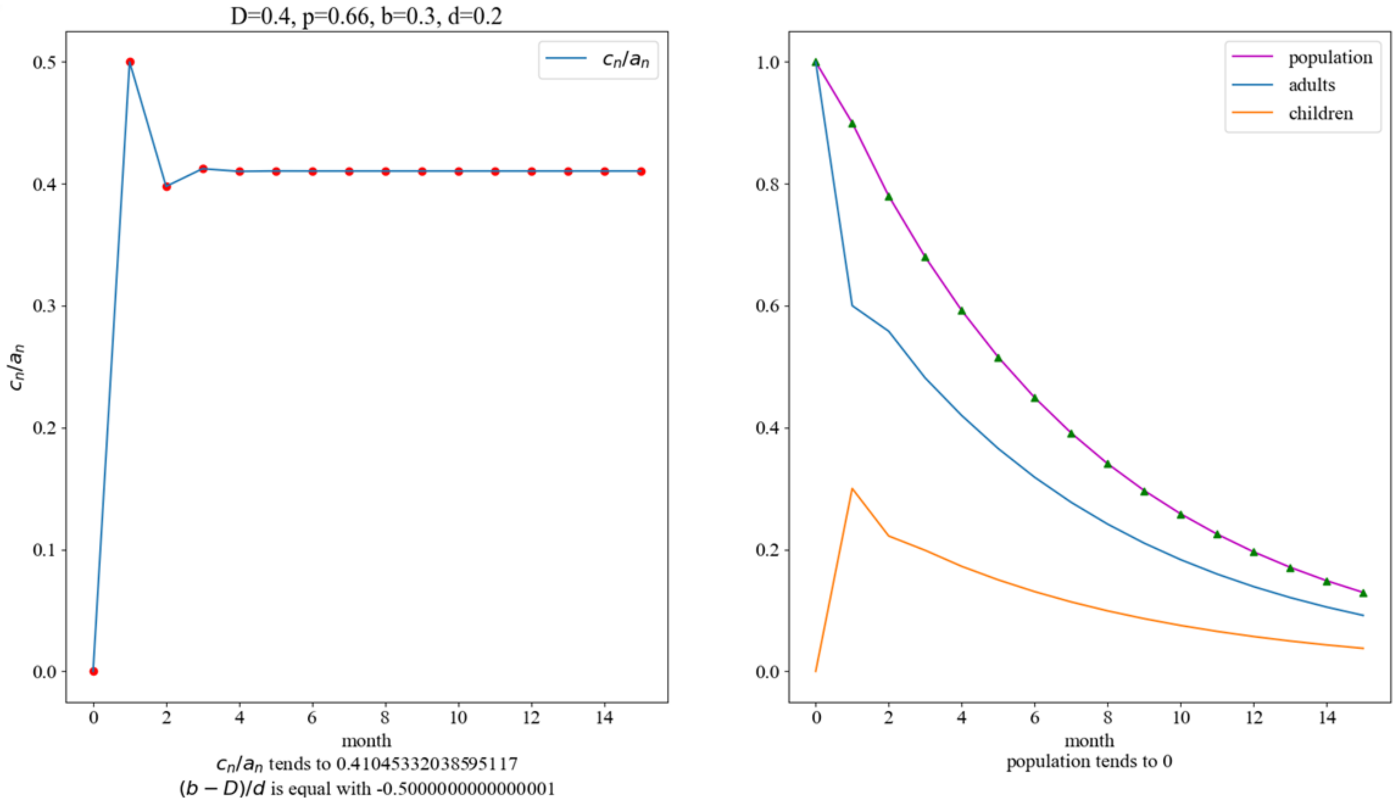

Figure 2. The ratio *children:adults* and the dynamics of the rabbit population for the case D = 0.4, p = 0.66, b = 0.3, d = 0.2

Case ***b* = 0.9, p = 0.6, d = 0.3, D = 0.6.** Here, $\mathbf{a_1 = 10, c_1 = 0}$.

We can observe that $c_n / a_n > (b-D)/d$, therefore the population will drop to 0.

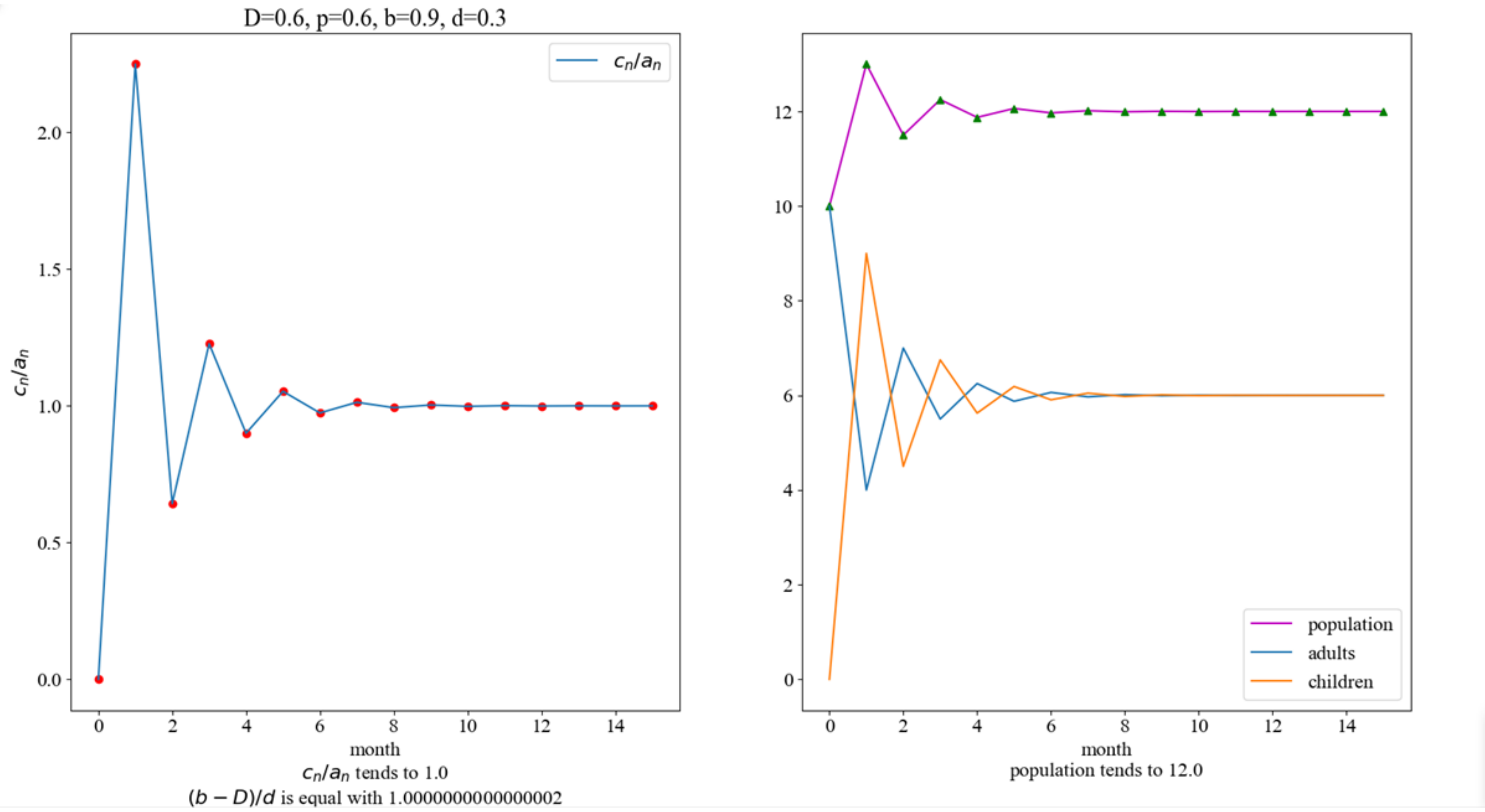

Figure 3. Case **b = 1.4, p = 0.66, d = 0.3, D = 0.7.** Here, $\mathbf{a_1 = 10, c_1 = 0}$.

We have $A = \begin{pmatrix} 0,4 & 0,6 \\ 0,9 & 0,1 \end{pmatrix}$ and $A^n = \begin{pmatrix} \dfrac{2\cdot(-1)^n + 3\cdot 2^n}{5\cdot 2^n} & \dfrac{-2\cdot(-1)^n}{5\cdot 2^n} \\ \dfrac{-3\cdot(-1)^n + 3\cdot 2^n}{5\cdot 2^n} & \dfrac{3\cdot(-1)^n + 2\cdot 2^n}{5\cdot 2^n} \end{pmatrix}$. We can see that the limit of $\dfrac{c_n}{a_n}$ is equal with $\dfrac{b-D}{d}$ (which is one), so the population will stabilize itself to a positive value (which is 12).

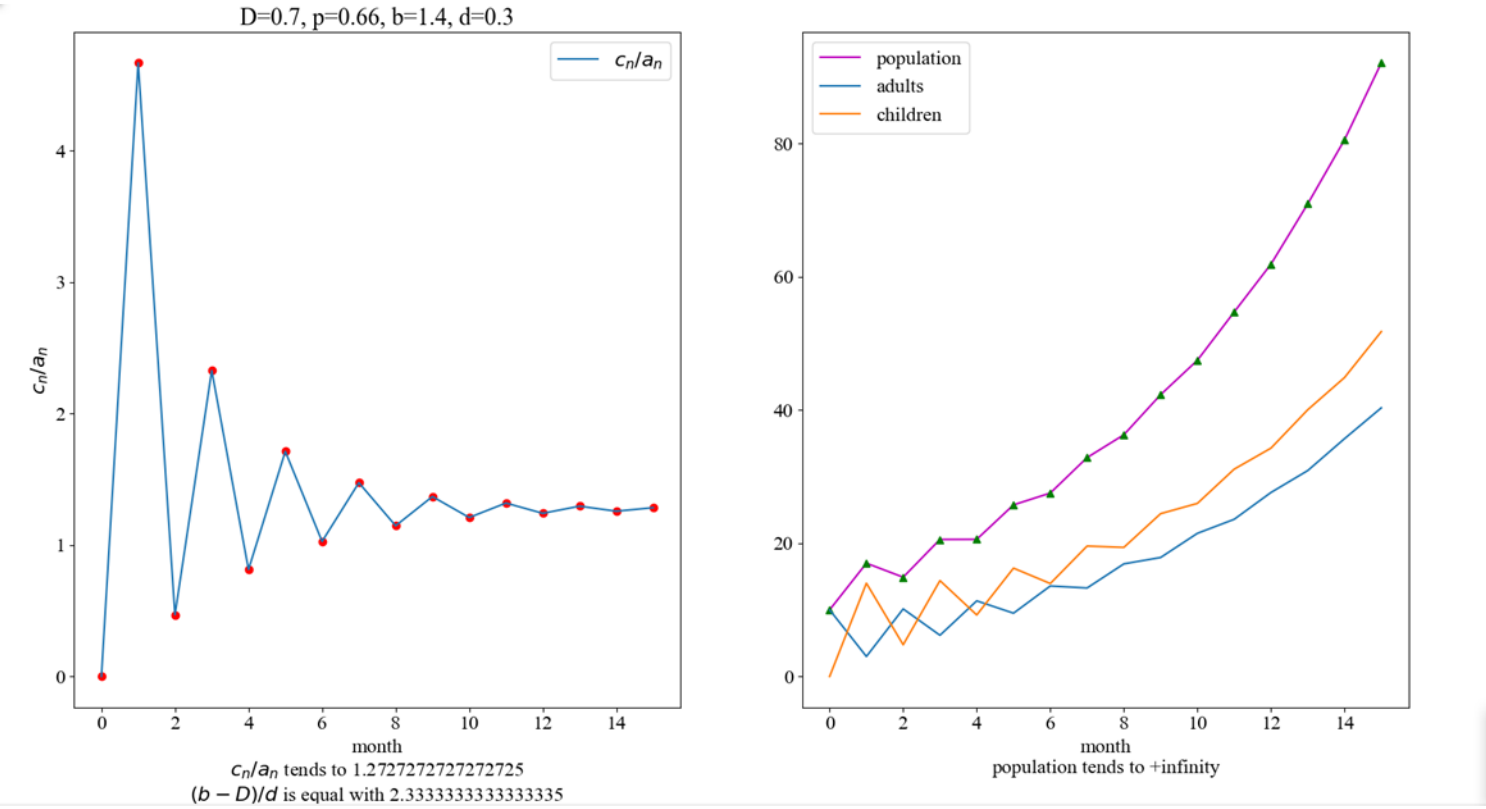

Figure 4. The ratio *children:adults* and the dynamics of the rabbit population for the case *D* = 0.7, p = 0.66, b = 1.4, d = 0.3. Here, we took $a_1 = 10$.

We have

$$A = \begin{pmatrix} 0,3 & 0,66 \\ 1,4 & 0,04 \end{pmatrix}$$

and

$$A^n = \begin{pmatrix} \dfrac{42\cdot(-1)^n\cdot 2^{3n}\cdot 5^n + 55\cdot 3^n\cdot 19^n}{97\cdot 2^n\cdot 5^{2n}} & \dfrac{-33\cdot(-1)^n\cdot 2^{3n}\cdot 5^n + 33\cdot 3^n\cdot 19^n}{97\cdot 2^n\cdot 5^{2n}} \\ \dfrac{-70\cdot(-1)^n\cdot 2^{3n}\cdot 5^n + 70\cdot 3^n\cdot 19^n}{97\cdot 2^n\cdot 5^{2n}} & \dfrac{55\cdot(-1)^n\cdot 2^{3n}\cdot 5^n + 42\cdot 3^n\cdot 19^n}{97\cdot 2^n\cdot 5^{2n}} \end{pmatrix}.$$

The limit of $\frac{c_n}{a_n}$ is smaller than $\frac{b-D}{d}$, so the population will grow to infinity.

4. CONCLUSION

In this paper, we considered a more realistic model for the evolution of a rabbit population than the Fibonacci model, in which we took into account the mortality rates for baby rabbits (d) and adult (D) rabbits, the breeding rate (b), as well as the maturation rate (p). The new model can be useful for predicting the number of rabbits after each month, and the child:adults ratio. By varying the above rates, the dynamics of the rabbit population might change dramatically, from extinction of the species to the infinitely growing numbers of rabbits.