A Sweet Problem

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Presentation of the research topic

A cake as a cuboid has to be served by 100 persons at a birthday party. The person who celebrates his birthday has to cut the cake in a way such that everybody who is there can eat a piece in order to taste the sweet cake. Pieces can have any shape or size; they do not have to be equal to each other, but they cannot be rearranged after any cut. The cake can be cut every direction.

a) What is the minimum number of cuts that the celebrated person should do in order to satisfy all guests (everybody tastes the cake)?

b) What is the minimum number of cuts in order to get equal pieces?

c) The cake is covered by chocolate (except the base). The area covered by chocolate is equal to 0.4 m^2. Find the dimensions of the cake that has maximum volume. The thickness of the glaze is disregarded.

d) A company produces packs for cakes. The pack has to be a cuboid made of cardboard with a volume of 0.03 m^3, with a double base as a square (two congruent squares, one over the other one). The price of cardboard is 0.5 USD for 1 m^2. Build the cheapest box.

Part a)

What is the minimum number of cuts the celebrated person should do in order to satisfy all guests (everybody tastes the cake)?

Solution

Number of cuts	0	1	2	3	4…
Number of pieces	1	2	4	8	15…

Below, we write the differences, the difference of differences and so on, for the consecutive values of a_n.

1 2 4 8 15

1 2 4 7

1 2 3

1 1

We observe that after three iterations we arrive at a constant sequence. [1] Therefore, according to the theory of recurrent sequences, the general term of the sequence will be of the form: [2]

$$a_n = an^3 + bn^2 + cn + d$$

Note: If a_4 would have been 16, then a_n would have a geometric form. The degree of a_n in our case is three as the next difference in the sequence of differences above is constantly 0.

By writing the first four terms of the sequence, we have:

$$a_0 = a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = 1 \Leftrightarrow d = 1$$

$$a_1 = a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d = 2 \Leftrightarrow a + b + c = 1$$

$$a_2 = a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d = 4 \Leftrightarrow 8a + 4b + 2c = 3$$

$$a_3 = a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d = 8 \Leftrightarrow 27a + 9b + 3c = 7$$

Therefore, we solve the system for *a, b, c, d* and we obtain:

$$a = \frac{1}{6},\ b = 0,\ c = \frac{5}{6}.$$

Then, $a_n = \frac{1}{6}n^3 + \frac{5}{6}n + 1.$

Calculating, we get :

$a_5 = 26,\ a_6 = 42,\ a_7 = 64,\ a_8 = 93,\ a_9 = 130.$

Thus, we need 9 cuts in order to have at least 100 pieces of cake.

Part b)

Whiat is the minimum number of cuts in order to get equal pieces? [3]

We found a rule for the number of cuts in order to get 100 equal pieces for the guests at that party.

For instance, we can cut the length of the cake into 5 equal horizontal columns (4 cuts), the width into 4 equal rows (3 cuts) and the height into 5 equal layers (4 cuts). Any other similar way of cutting will lead us to 100 equal pieces of cake.

Therefore, we need to cut the cake 11 times in the way mentioned before in order to get 100 equal pieces of cake.

Part c)

The cake is covered by chocolate (except the base). The area covered by chocolate is equal to 0.4 m^2. Find the dimensions of the cake that has maximum volume. The thickness of the glaze is disregarded

We denote by *a* the length of the cake, by *b* the width of the cuboid cake, and by *c* the height of it. The volume of this cuboid is

$$V = a \cdot b \cdot c$$

and the area covered by chocolate is equal to $A = a \cdot b + 2a \cdot c + 2b \cdot c$. In order to get the value of dimensions of the cake for maximum volume when the area is given (**0.4 m^2**), we use the inequality of mean (the arithmetic mean is greater than or equal to geometric mean). [4]

More precisely,

$$A=\frac{a\cdot b}{2}+\frac{a\cdot b}{2}+a\cdot c+a\cdot c+b\cdot c+b\cdot c=2(\frac{a\cdot b}{2}+a\cdot c+b\cdot c).$$

$$A\geq 2\cdot 3\cdot\sqrt[3]{\frac{a\cdot b}{2}\cdot a\cdot c\cdot b\cdot c}=\frac{6}{\sqrt[3]{2}}(a\cdot b\cdot c)^{2/3}=3\sqrt[3]{4}\cdot V^{2/3}.$$

$$0,4\geq 3\sqrt[3]{4}\cdot V^{2/3}.$$

Therefore, the maximum value of volume is equal to

$$V=\frac{4}{30\sqrt{30}}\ \text{m}^3$$

and it is reached when $a=b=2c$.

Thus, $a=b=\frac{\sqrt{30}}{15}$ m and $c=\frac{\sqrt{30}}{30}$ m, which means, approximating, $a=b=36.5$ cm and $c=18,25$ cm.

Part d)

A company produces packs for cakes. The pack has to be a cuboid made of cardboard with a volume of 0.03 m^3, with a double square as a base (two congruent squares, one over the other one). The price of cardboard is 0.5 USD for 1 m^2. Build the cheapest box.

We will split the problem into two cases:

I. The base is a square.

II. The base is a double square (two congruent squares, one over the other one).

I. The base is a single square with.

We assume, in this case, that $a=b$. This means that the area is equal to $A=2a^2+4a\cdot c$ and the volume is equal to $V=a^2\cdot c=0.03\,\text{m}^3$.

Using same technique as in c), we get

$$A=2a\cdot c+2a\cdot c+2a^2\geq 3\cdot\sqrt[3]{2a\cdot c\cdot 2a\cdot c\cdot 2a^2}=6\cdot V^{2/3}.$$

The minimum area is reached when $a=c$, which means that $a^3=0.03\Leftrightarrow a=\sqrt[3]{0.03}\,\text{m}$. Thus, the area is $A=6\cdot V^{2/3}=6\cdot 0.03^{2/3}$ m^2, which means approximately $A\approx 0.579$ m^2. Taking into consideration the price for 1 m^2 of cardboard, we obtain the price of the cheapest box, more precisely about $0.579\cdot 0.5\approx 0.29$ USD, which means 29 cents.

II. The base is a double square (two congruent squares, one over the other one).

The area is equal to $A=3a^2+4a\cdot c$ and the volume is equal to $V=a^2\cdot c=0.03\,\text{m}^3$.

Using same technique as in c), we get

$$A=2a\cdot c+2a\cdot c+3a^2\geq 3\cdot\sqrt[3]{2a\cdot c\cdot 2a\cdot c\cdot 3a^2}=3\cdot\sqrt[3]{12}\cdot V^{2/3}.$$

The minimum area is reached when $c=\frac{3a}{2}$, which means that $a^3=0.02\Leftrightarrow a=\sqrt[3]{0.02}$ m. Thus, the area is $A=9\cdot a^2=9\cdot 0.02^{2/3}$ m^2. Taking into consideration the price for 1 m^2 of cardboard, we obtain the price of the cheapest box in this case, more precisely about $9\cdot 0.02^{2/3}\cdot 0.5=4.5\cdot 0.02^{2/3}$ USD.

Edition Notes

[1] Strictly speaking, we are not sure that the final sequence is constant, but we can be confident about this. The result should be proved by induction.

[2] Here, the general result that is used should be mentioned explicitly.

[3] It should be said explicitly that this problem is solved by attempts. It should be also observed that the maximum number of parts, and hence the minimum number of cuts, is obtained when the numbers of cuts in different directions are near.

[4] It is true that "the arithmetic mean is greater or equal to the geometric mean". It should be said explicitly, though, that the proof of this is easy only in the case of two numbers.