Tile Wallpaper

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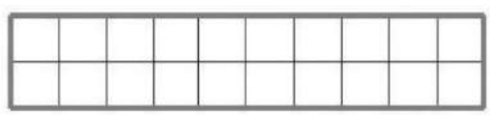
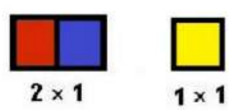
The problem (Abstract)

A rectangular wall of size 2×10 is to be covered with tiles. The number of tiles is unlimited and each tile can be rotated according to your preference before mounting. In how many ways can we cover the entire wall using only 2×1 coloured tiles? What if we have access to both types of tiles, 1×1 and 2×1?

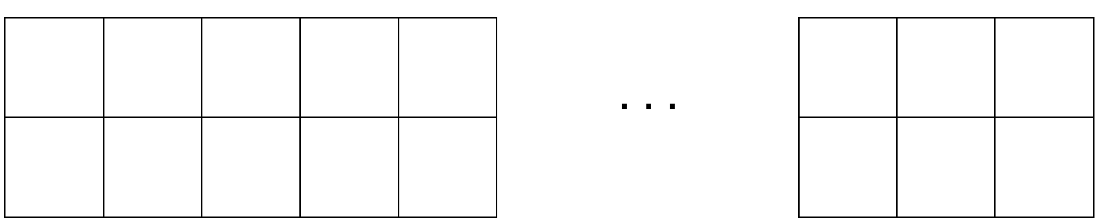




For **the first part** of our problem, we only have 2×1 tiles available, therefore we will analyze two cases: the first one, in which the tiles are uncoloured, and the second one, in which the tiles are coloured.

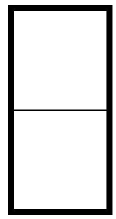
Case I: Uncoloured tiles

We are looking for a way of placing the uncoloured tiles. To be able to find a recurrent formula, we consider a $2 \times n$ wall, then we particularize for $n = 10$.



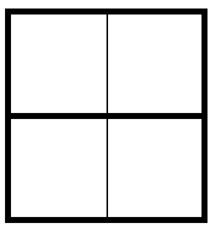
n columns

We denote by a_n the number of ways we can cover the $2 \times n$ wall with 2×1 uncoloured tiles.
Firstly we count the number of ways we can cover the last column. We see the last column can be covered in only **one way**, as displayed below.

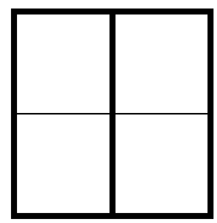


The other $(n-1)$ columns will be covered in a_{n-1} ways.

We then count the number of ways we can cover the last two columns. We see that the last two columns can be covered in two ways (as displayed below), but one of these two cases was considered before, when the last column was covered (the second one). So, there is only **one way** remaining, the first one.



or



The other $(n-2)$ columns will be covered in a_{n-2} ways. [1]

If we consider the covering of the last three or more columns, we will not discover new ways that were not considered before. We arrive at the following recurrence relation:

$$(1)\ a_n = a_{n-1} + a_{n-2}, \text{ for all } n \geq 3.$$

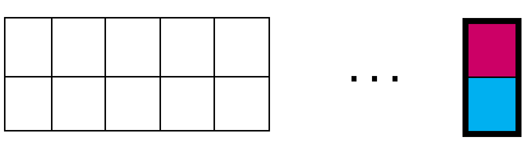
The first two terms of the sequence a_n are

$$a_1 = 1; a_2 = 2.$$

Note that relation (1) is the recurrence for the Fibonacci numbers.

Case II: coloured tiles

We denote by b_n the number of ways we can cover the $2 \times n$ wall with 2×1 coloured tiles. We firstly count the number of ways we can cover the last column. We see the last column can covered in **two ways**, as displayed below.



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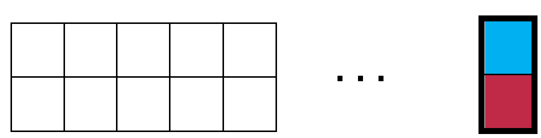
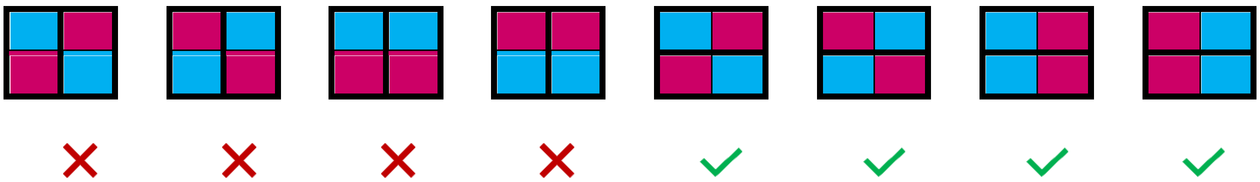


The other $(n-1)$ columns will be covered in b_{n-1} ways.

We then count the number of ways we can cover the last two columns. We see that the last two columns can be covered in eight ways (as displayed below). However, four of these eight cases (with the tiles places vertically) were considered in before, when the last column was covered, and we disregard them.

So, there are only **four ways** remaining, the ones marked with green in the figure below (with the tiles placed horizontally).



The other (n– 2) columns will be covered in b_{n-2} ways.

If we consider the covering of the last three or more columns, we will not discover new ways that were not considered before. We arrive at the following recurrence relation:

$$(2)\ b_n = 2b_{n-1} + 4b_{n-2}\,, \qquad \text{for all} n \geq 3.$$

Also note that we can arrive at recurrence (2) in a different way, as follows. As each square in a domino can be painted in two colours [2], we get that

$$b_n = 2^n \cdot a_n, \text{for all} n \geq 1.$$

Then, we multiply recurrence (1) by 2^n, we arrive at

$$b_n = 2^n \cdot (a_{n-1} + a_{n-2}), \quad \text{for all } n \geq 3,$$
$$b_n = 2^n \cdot a_{n-1} + 2^n \cdot a_{n-2}, \text{ for all } n \geq 3.$$

We want to determine a recurrence relation for the (b_n).

Because $b_{n-1} = 2^{n-1} \cdot a_{n-1}$ and $b_{n-2} = 2^{n-2} \cdot a_{n-2}$, then

$$b_n = 2 \cdot 2^{n-1} \cdot a_{n-1} + 4 \cdot 2^{n-2} \cdot a_{n-2}, \text{for all } n \geq 3,$$

$$b_n = 2b_{n-1} + 4b_{n-2}, \text{ for all } n \geq 3.$$

The first two terms of the sequence b_n are

$$b_1 = 2;\ b_2 = 8.$$

We then have successively

$$b_3 = 2b_2 + 4b_1 = 2 \cdot 8 + 4 \cdot 2 = 24,$$
$$\dots$$
$$\dots$$
$$\dots$$
$$b_{10} = 2b_9 + 4b_8 = 2 \cdot 28160 + 4 \cdot 8704 = 91136.$$

Thus, there are **91136** ways of covering the $2 \times n$ wall with coloured dominos.

We can also determine this number by finding the general formula of the recurrence (2). We will proceed as follows.

The characteristic equation attached to recurrence $b_n = 2b_{n-1} + 4b_{n-2}$ is

$$\alpha^2 = 2\alpha + 4.$$

Seeing that b_n is given by a second order recurrence relation, it will take the following form:

$$b_n = C_1\alpha_1^n + C_2\alpha_2^n$$

where $\alpha_1$andα_2 represent the solutions of the equation:

$$\alpha_1 = 1 + \sqrt{5},$$
$$\alpha_2 = 1 - \sqrt{5}.$$

We determine C_1 and C_2:

For $n = 1$, $b_1 = C_1 \cdot \alpha_1 + C_2 \cdot \alpha_2$, then $2 = C_1(1+\sqrt{5}) + C_2(1-\sqrt{5})$;

For $n = 2$, $b_2 = C_1 \cdot \alpha_1^2 + C_2 \cdot \alpha_2^2$, then $8 = C_1(1+2\sqrt{5}+5) + C_2(1-2\sqrt{5}+5)$.

We form the following system:

$$\begin{cases} 2 = C_1(1+\sqrt{5}) + C_2(1-\sqrt{5}) \\ 8 = C_1(6+2\sqrt{5}) + C_2(6-2\sqrt{5}), \end{cases}$$

$$\begin{cases} 2 = C_1(1+\sqrt{5}) + C_2(1-\sqrt{5}) \\ 4 = C_1(3+\sqrt{5}) + C_2(3-\sqrt{5}), \end{cases}$$

which we will solve using the reduction method:

$$C_1 = \frac{\sqrt{5}+5}{10},$$
$$C_2 = \frac{-\sqrt{5}+5}{10}.$$

We obtain $b_n = \frac{\sqrt{5}+5}{10}(1+\sqrt{5})^n + \frac{-\sqrt{5}+5}{10}(1-\sqrt{5})^n$

$$\boxed{b_n = \frac{\sqrt{5}}{10}(1+\sqrt{5})^{n+1} - \frac{\sqrt{5}}{10}(1-\sqrt{5})^{n+1}}$$

The general term formula

To help calculate b_{10} we shall use the Newton binomial:

$$(a+b)^n = a^n + C_n^1 \cdot a^{n-1} \cdot b + C_n^2 \cdot a^{n-2} \cdot b^2 + \cdots + b^n.$$

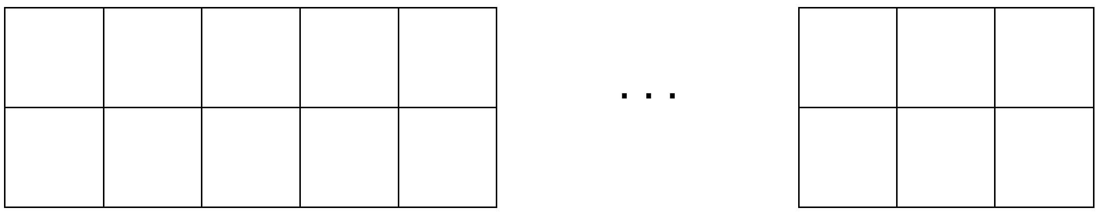
Now we can find out the number of combinations needed:

$$\begin{aligned} b_{10} &= \tfrac{\sqrt{5}}{10}(1+\sqrt{5})^{11} - \tfrac{\sqrt{5}}{10}(1-\sqrt{5})^{11} b_{10} \\ &= \tfrac{\sqrt{5}}{10}\left[(1+\sqrt{5})^{11} - (1-\sqrt{5})^{11}\right] \\ &= \frac{\sqrt{5}}{10}\left(2C_{11}^1 \cdot \sqrt{5} + 2C_{11}^3 \cdot \sqrt{5}^3 + \cdots + 2\sqrt{5}^{11}\right) \\ &= 91136. \end{aligned}$$

After calculations we get that the number of combinations of 2×1 colored tiles on a 2×10 wall is 91136.

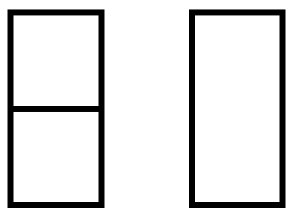
For **the second part** of our problem both types of tiles are available to us, therefore we will work with 2 cases again, one for uncoloured tiles and one for coloured tiles.

Case I: Uncoloured tiles



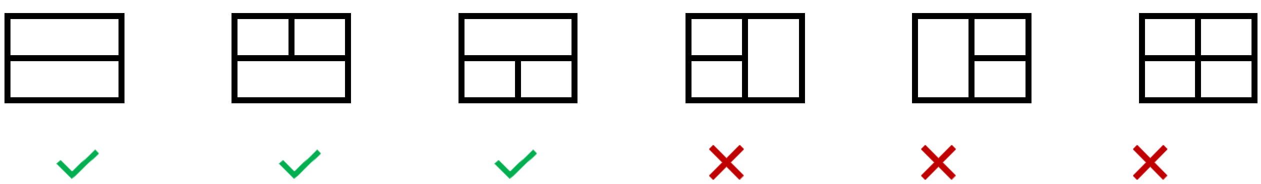
n columns

We consider a $2 \times n$ wall, then we will particularize for $n = 10$. We denote by c_n the number of ways we can cover the 2×10 wall with both types of uncolored tiles.
We count the number of ways we can cover the last column, same as we did in the first part of the problem. The following 2 cases are observed:



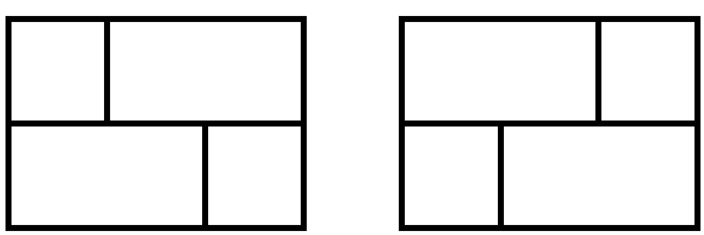
The other $(n - 1)$ columns will be covered in c_{n-1} ways.

We can see that the last two columns can be covered in six ways (as displayed below). However, three of the ways (the ones marked with the X) were considered when we covered the last column. Therefore, there will be only three ways of covering the columns (the ones with the tiles placed horizontally).



The remaining $(n - 2)$ columns can be covered in c_{n-2}ways.

For covering the last 3 columns, we have only 2 new ways:



For each of the last k columns ($k \geq 4$), there are exactly 2 ways to cover the extra column, a new way and also its opposite: [3]

$$2c_{n-4} + 2c_{n-5} + \cdots + 2c_1 + 2$$

We now write the following relation:

$$c_n = 2c_{n-1} + 3c_{n-2} + 2c_{n-3} + 2c_{n-4} + 2c_{n-5} + \cdots + 2c_1 + 2$$

But

$$c_{n+1} = 2c_n + 3c_{n-1} + 2c_{n-2} + 2c_{n-3} + 2c_{n-5} + \cdots + 2c_1 + 2.$$

We subtract the two relations, remaining with

$$c_{n+1} - c_n = 2c_n + c_{n-1} - c_{n-2}.$$

We add c_n on both sides, getting to the relation written below:

$$\boxed{c_{n+1} = 3c_n + c_{n-1} - c_{n-2} \qquad \text{for all } n \geq 3.}$$

The first three terms of the c_n sequence are

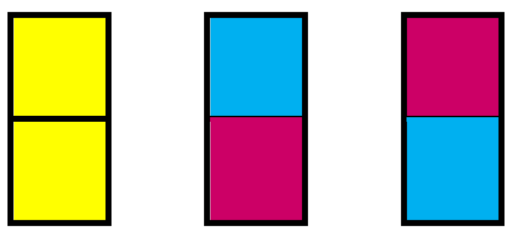
$$c_1 = 2, c_2 = 7, c_3 = 22.$$

$c_{10} = 78243$ ways of covering the 2×10 wall with 2×1 and 1×1 uncoloured tiles.

In conclusion, there are 78243 ways of covering the 2×10 wall with both 2×1 and 1×1 uncoloured dominos.

Case II: coloured tiles

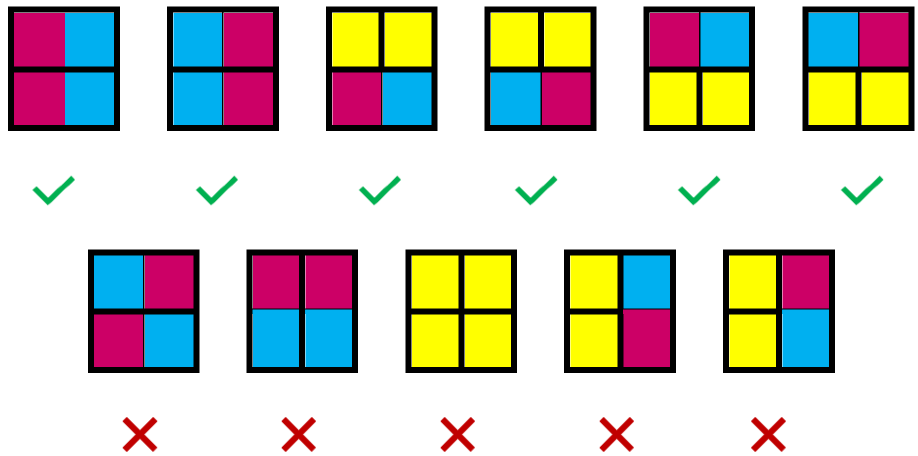
For each way of placing the uncoloured tiles, we have to identify how many different situations can appear if they get coloured in the specified colours.
For the last column in the $2 \times n$ wall, we have already discovered two ways of placing the uncoloured tiles. If we color them, three ways will now be valid, since we can rotate the 2×1 dominos.



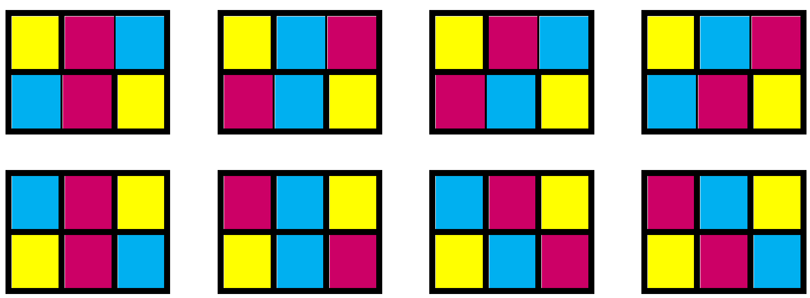
The other $(n-1)$ columns can be covered in d_{n-1} ways.

For the last two columns, there are 3 new ways of placing the uncoloured tiles, and 6 new ways of placing the coloured tiles [4]. In this case the only valid versions will be the ones in which the 2×1 tiles are placed horizontally, because the cases in which the tiles are placed vertically have already been counted before, when we covered the last column.

The other $(n - 2)$ columns can be covered in d_{n-2} ways.



For the last three columns, there are 2 new ways of placing the uncoloured tiles and 8 new ways of placing the coloured tiles:
The other $(n-3)$ columns can be covered in d_{n-3} ways:



Thus, we write the following mathematical relation: [5]

$$d_n = 3d_{n-1} + 6d_{n-2} + 8d_{n-3} + 8d_{n-4} + \cdots + 8d_2 + d_1.$$

For $(n+1)$ columns:

$$d_{n+1} = 3d_n + 6d_{n-1} + 8d_{n-2} + 8d_{n-3} + \cdots + 8d_2 + d_1.$$

If we subtract the two relations we get

$$d_{n+1} = 4d_n + 3d_{n-1} + 2d_{n-2} (n \geq 3).$$

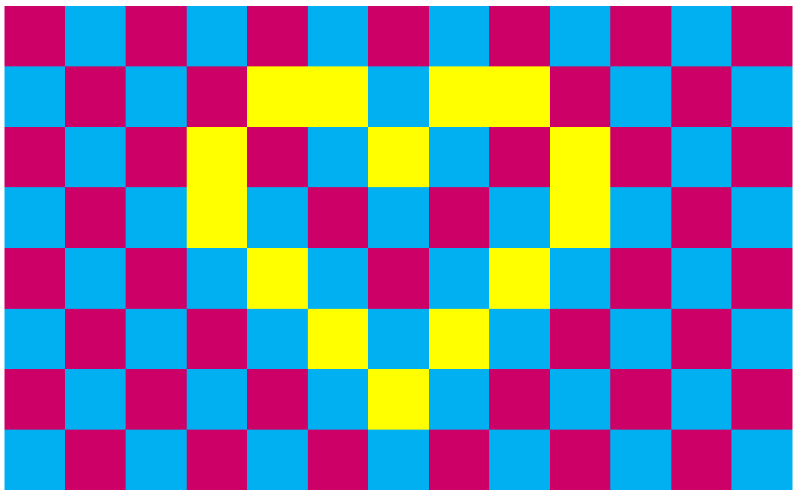
The terms of the d_n sequence are

$d_1 = 3;$
$d_2 = 17;$
$d_3 = 83;$
$d_4 = 4d_3 + 3d_2 + 2d_1 = 4 \cdot 83 + 3 \cdot 17 + 2 \cdot 3 = 389;$
$d_5 = 4d_4 + 3d_3 + 2d_2 = 155 + 249 + 34 = 1839;$
...
...
...
$d_{10} = 4d_9 + 3d_8 + 2d_7 = 366538 + 581847 + 82102 = 4329257.$

$d_{10} = 4.329.257$ ways of covering the 2×10 wall with both types of coloured tiles.

Conclusion

Considering the multitude of possibilities, we realized from the beginning that we must be thorough in our calculations. First of all, we considered two main cases: the usage of uncoloured and coloured tiles. Then we realized that we have to discover a general formula to apply to our case. Along the way, the problem became more and more complex. But it was easier when we applied the formula to the cases, finally managing to find out the situations and fulfill our goal.



References

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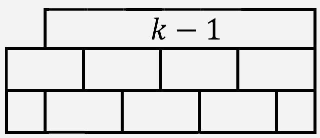
- https://www.cuemath.com/data/combinations/

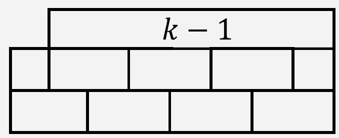
EDITION NOTES

[1] This explanation of formula (1) is a bit confused. A better explanation would be the following. Two possible cases can be considered: either the last tile is vertical or not. There are a_{n-1} coverings in which the first case holds. If the last tile is not vertical, then the last two tiles are horizontal, and hence there are a_{n-2} coverings in which the second case holds.

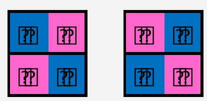
[2] It would be better to be more precise. "For each covering with uncolored tiles, every tile can be colored in two different ways".

[3] The sentence "For each of the last k columns ...." is rather obscure. It is not easy to interpret it. It should be firstly said that the main point is to find the tilings of the first k columns that are not extensions of tilings of the first h columns for some $h < k$. Then, using following pictures as an example, one can conclude that there are 2 tilings of the first k columns with that property.





[4] The new ways are 8. There are two possibilities missing in the figure:



[5] As observed in the previous note, the coefficient of d_{n-2} in the formula for d_n (and of d_{n-1} in that for d_{n+1}) is 8, not 6. There is also an error in the number of ways to colour the $2 \times k$ rectangles. Since the number of 1×2 tiles is $k-1$, and each tile can be coloured in 2 ways, there are 2^{k-1} possibilities for each of the two cases considered in Note [3]. Then, we should have $2^k d_{n-k}$ instead of $8\, d_{n-k}$ in the expressions for d_n and d_{n+1}.
All the formulas at the end of the work have to be fixed according to these observations.