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Patterns in friezes

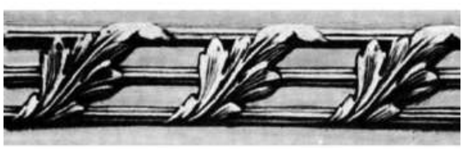
2024

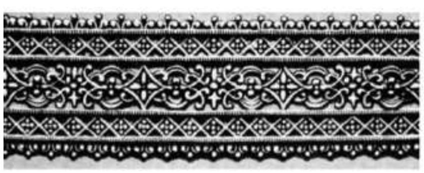
I.1. Definition

A **frieze group** is an interesting collection of infinite symmetry groups that arise from periodic designs in a plane.

In mathematics, a **frieze** or **frieze group** is a two-dimensional design that repeats in one direction. Such patterns occur frequently in architecture and decorative art.

Examples:





To generate a frieze pattern we identified a set of operations:

I.2 Translation

I.2.1 Definition: A translation is a geometric transformation that moves every point of a figure, shape or space by the same distance in a given direction.
In a two dimensional space a translation can also be interpreted as the addition of a constant vector to every point, or as shifting the origin of the coordinate system.

I.2.2 The function

If $\vec{v}$is a fixed vector, named the translation vector, and P is the initial position of some object, then the translation function T will be defined:

$$T: \mathbb{P} \to \mathbb{P}$$
$$T(A) = A' \Leftrightarrow \overrightarrow{AA'} = \vec{v}$$

I.2.3 Example:

Source: Greek temple

I.3 Symmetry

I.3.1 Definition: A symmetry is a mapping from a Euclidean space to itself that is an isometry with a hyperplane as a set of fixed points; this set is called the axis (in dimension 2) or plane (in dimension 3) of reflection.

Remark 1: We will only analyze the symetrys in a two-dimensional space, taking the X-axis and then the Y-axis of a cartezian system(in which the frieze is situated) as the axis of reflection.

I.3.2 The function

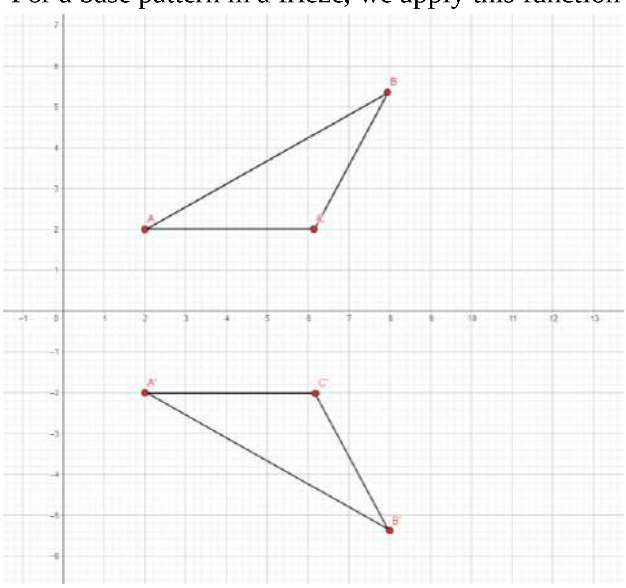
i) Symmetry with the X-axis as a reflection axis

Let the function be:

$S_{ox}: \mathbb{P} \rightarrow \mathbb{P}$

$S_{ox}(A) = A' \Leftrightarrow A(x, y) \Leftrightarrow A'(x, -y)$

For a base pattern in a frieze, we apply this function to every point of the object like such:



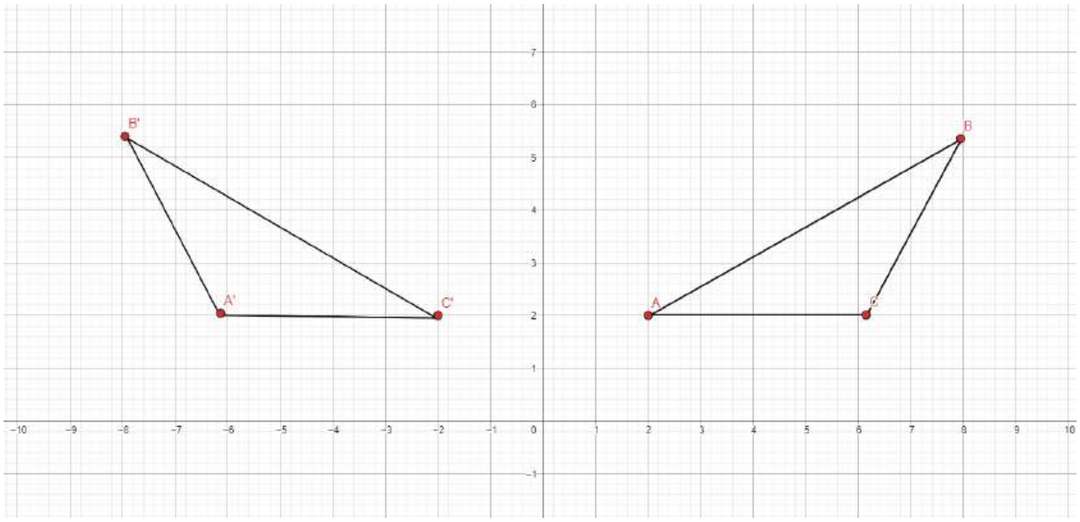
ii) Symmetry taking the Y-axis as a reflection axis

Let the function be:

$$S_{oy}: \mathbb{P} \rightarrow \mathbb{P}$$

$$S_{oy}(A) = A' \Leftrightarrow A(x,y) = A'(-x,y)$$

For a base pattern in a frieze, we apply this function to every point of the object like such:



I.3.3 Example:

Source: Greek temple

I.4 <u>Glide reflection</u>

I.4.1 Definition: A glide reflection or transflection (in a two dimensional space) is a geometric transformation that consists of a reflection across a line and a translation ("glide") in a direction parallel to that line, combined into a single transformation.

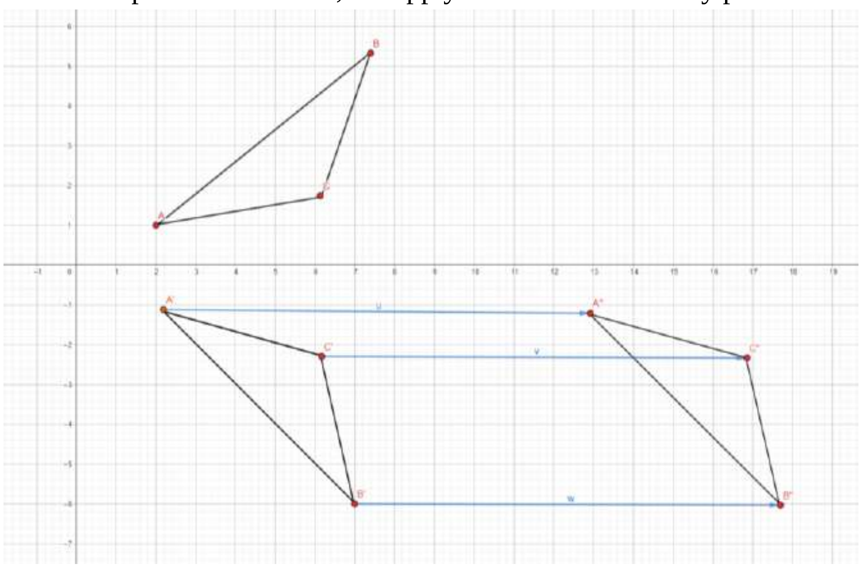
I.4.2 The function

Let the function be:

$Gr : \mathbb{P} \rightarrow \mathbb{P}$

$GR(A) = T(R_{ox}(A)) = A''$

For a base pattern in a frieze, we apply this function to every point of the object like such:



I.5 <u>Rotation</u>

I.5.1 Definition: A rotation is a transformation in which the object is rotated about a fixed point with a respective angle.

Remark 2: The direction of rotation can be clockwise or anticlockwise. The fixed point in which the rotation takes place is called the center of rotation. We will take the center of rotation the center of the base pattern and the direction anticlockwise.

I.5.2 The function

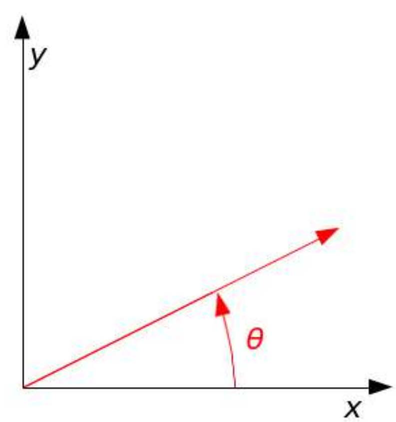
Let the function be:

$$R_\theta : \mathbb{P} \rightarrow \mathbb{P}$$

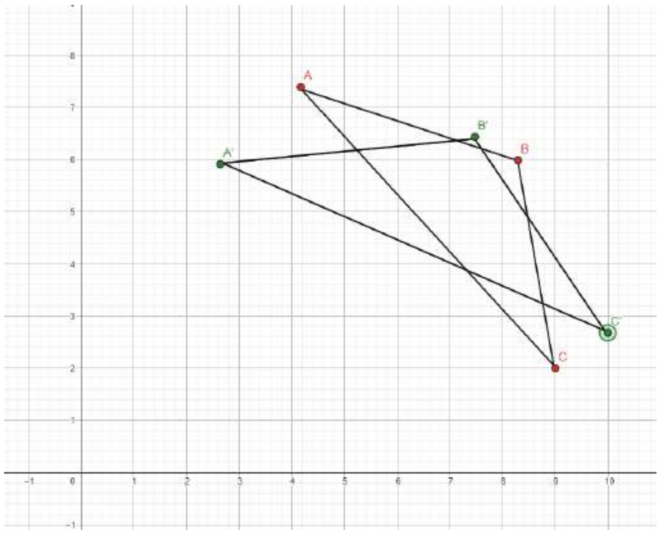
$$R_\theta(A) = A' \Leftrightarrow A(x, y) \text{ and } A'(x, y) \text{ where}$$

$$x' = x \cos\theta - y \sin\theta$$

$$y' = x \sin\theta + y \cos\theta$$



For a base pattern in a frieze, we apply this function to every point of the object like such to rotate the object by θ degrees like such:



Remark 3: In linear algebra, a **rotation matrix** is a transformation matrix that is used to perform a rotation in Euclidean space.

For example, using the convention below, the matrix rotates points in the *xy* plane counterclockwise through an angle θ about the origin of a two-dimensional Cartesian coordinate system.

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

To perform the rotation on a plane point with standard coordinates $\mathbf{v} = (x, y)$, it should be written as a column vector, and multiplied by the matrix R:

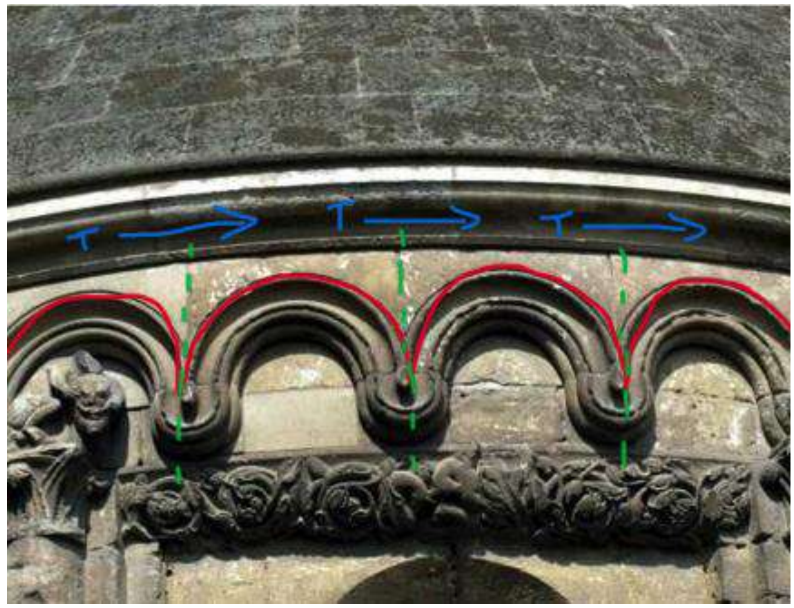
$$R\mathbf{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

Let V be the column vector which is composed of the x and y coordinates of the point we want to rotate. We will multiply it by the matrix

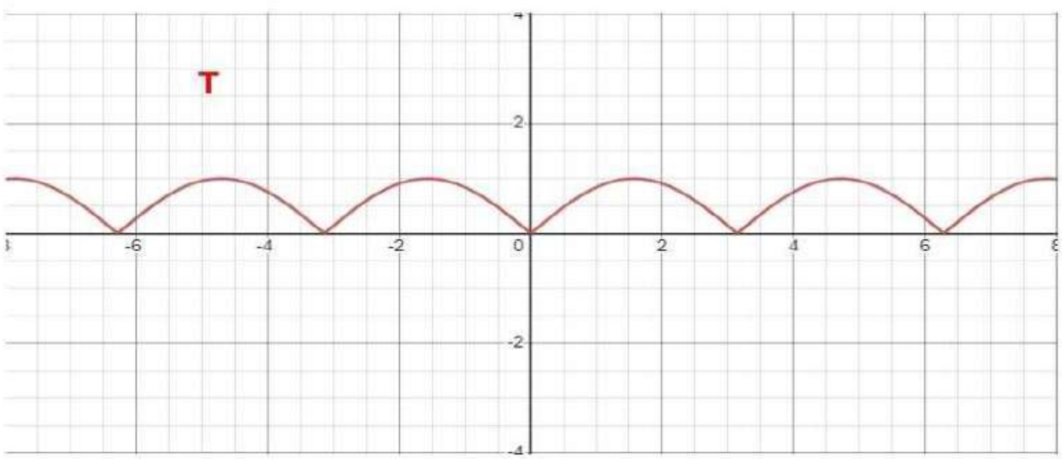
The product of these two matrices will be the column vector of the coordinates of the final point V'. Thus, the new coordinates (x', y') of a point (x, y) after rotation are:

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

II. Analyzing different friezes

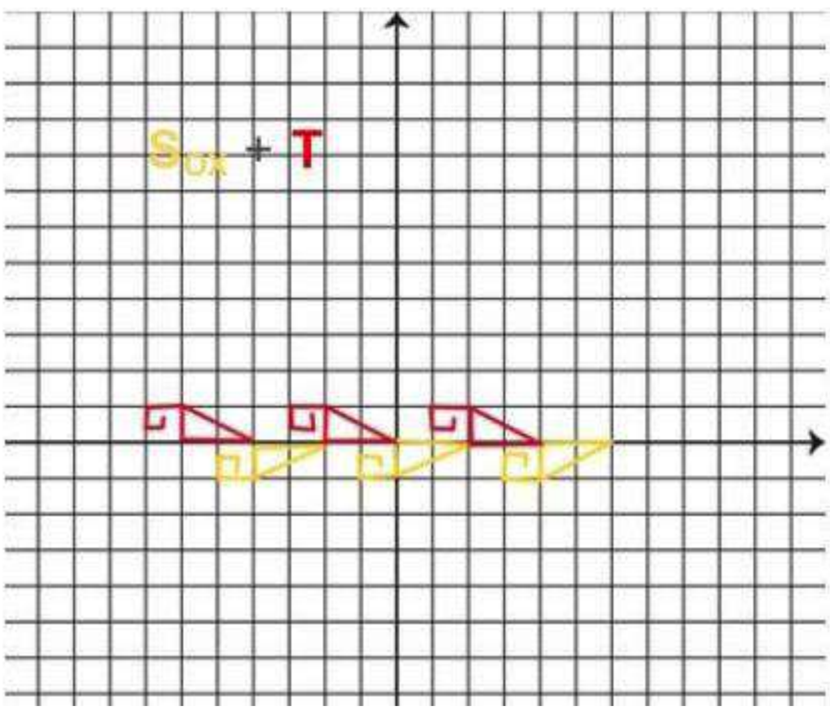


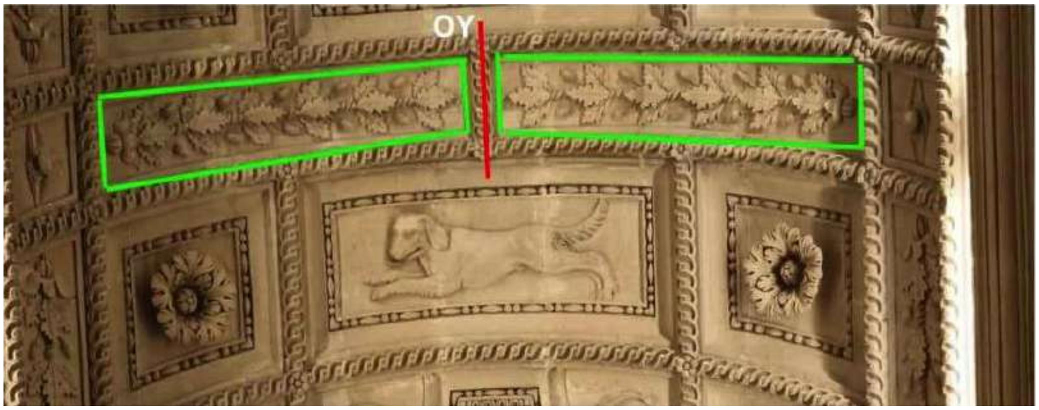
The frieze above is observed on a Greek temple and adheres to the criteria of translation: a pattern that repeats to infinity.

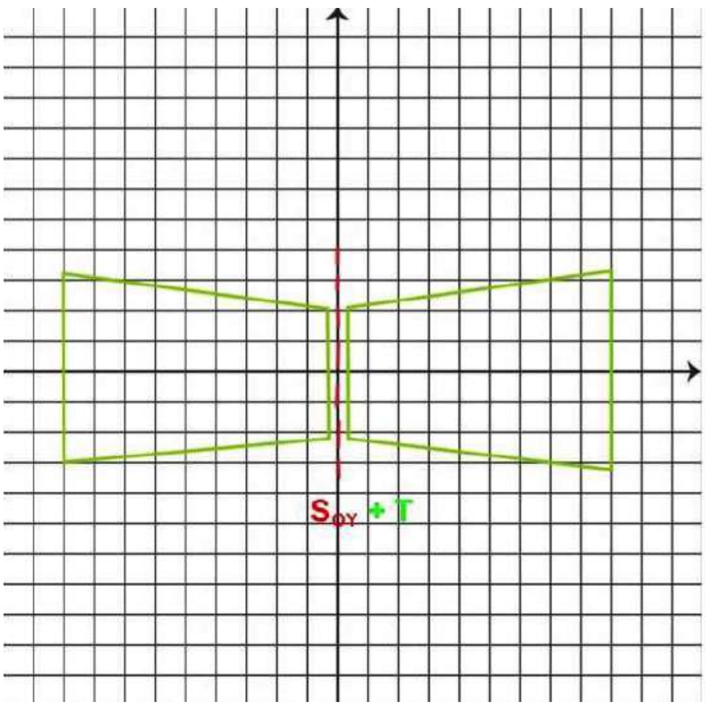


In a Cartesian system, we can observe that the pattern is similar to the graph of the modulus of sin X.

The frieze above is observed on an Aztec temple and is quite complex. The initial pattern includes a symmetry along the OX axis and then a translation to compose the final frieze.

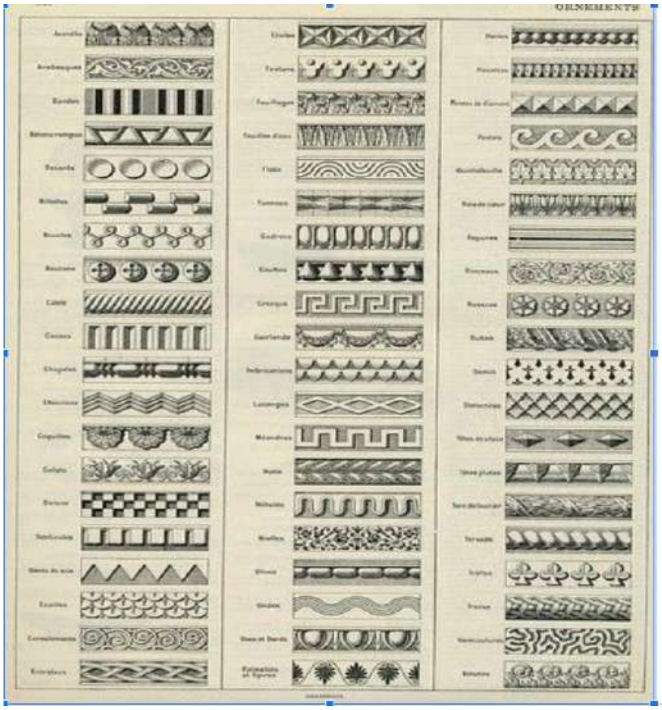






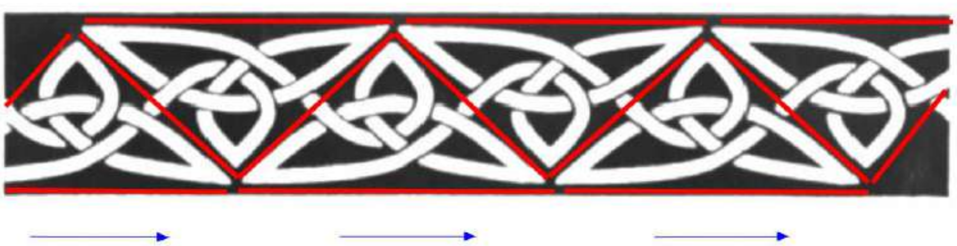
The frieze above is from the Louvre Museum and combines symmetry with translation. The initial arrangement is reflected on the OY axis and then translated to compose the final pattern.

This frieze is a decoration in greek temples, also known as meander due to its repeating S-like shape. This is a result of a symmetry on the OY axis of the half of the base model, that is then rotated upside down and translated.

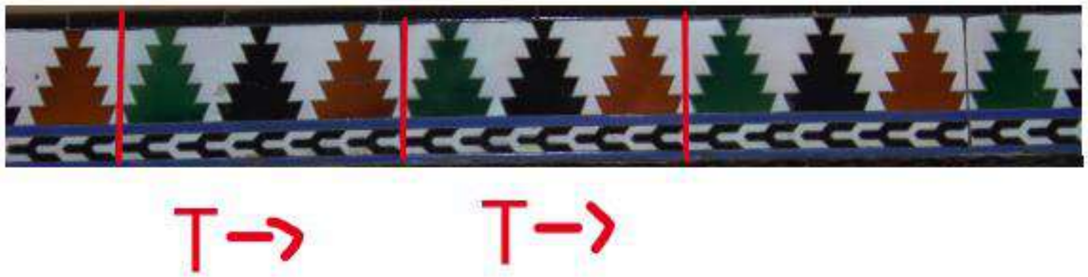


Ornaments. Stampa 1954 - Libro Usato - Larousse

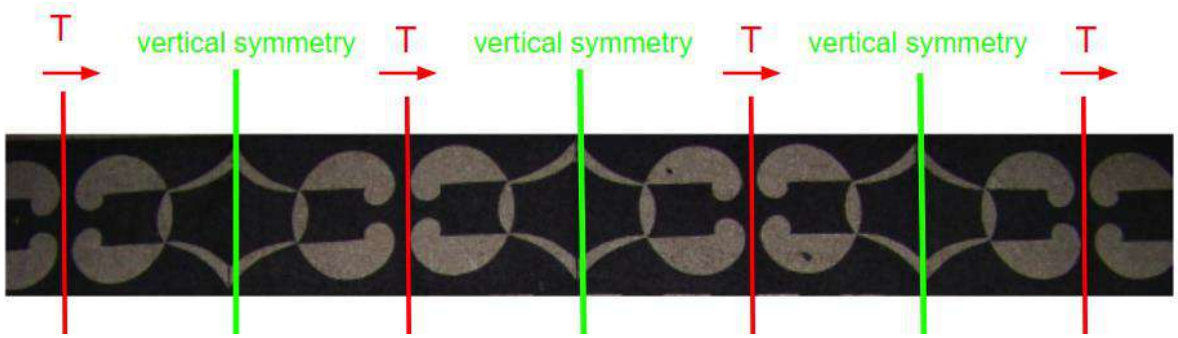
T→ + R

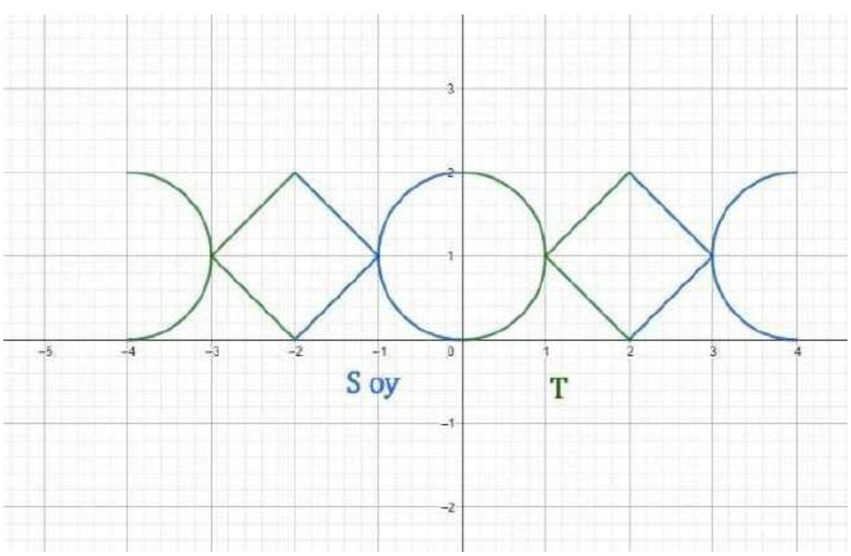


This Celtic frieze is obtained by rotating the base model by 180 degrees and then translating it, giving the model a continuous look.

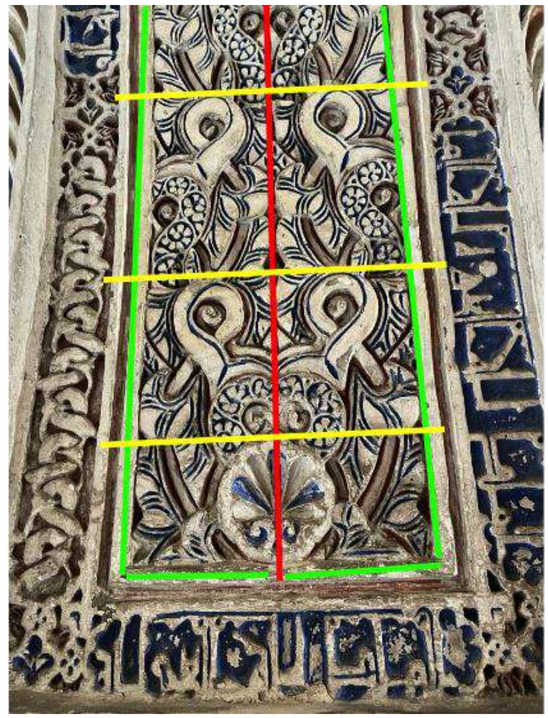


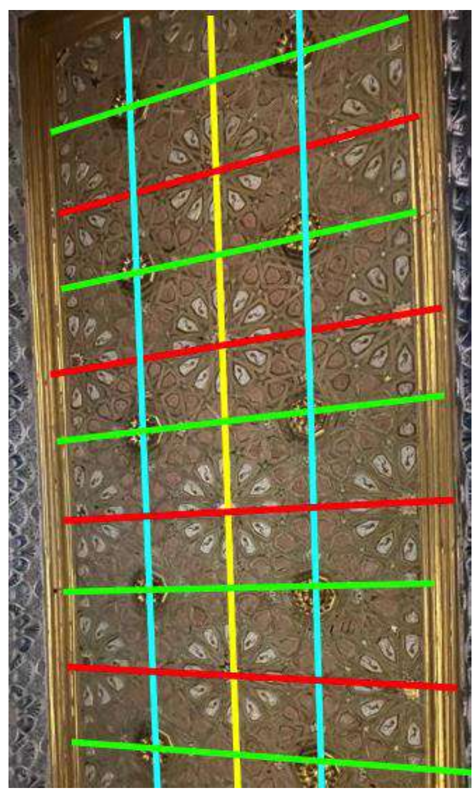
This frieze from the Richmond Landmark Theatre is a simple one, containing only translations to the right of the base model.





The frieze above can also be observed at the same location as the previous one, but it has a more complex base model. It is made by reflections on the OY axis of the base model and translations to the right, resulting in a pattern that we simplified in a cartesian representation.





III. How many types of friezes are there?

The variety of Frieze groups arises from the different ways in which translational, reflective, glide symmetries and rotation can be combined, giving rise to unique patterns.

Without the translation movement that is present in all friezes, there are 4 operations left. Intuitively, four transformations would produce $2^4=16$ patterns, but there are less than 16 because the result of some operators can be achieved through a combination of 2 others. For example, glide reflection is a combination of horizontal symmetry and translation. Also, a vertical symmetry and rotation are equivalent to horizontal symmetry.

Why are there only 7 types of friezes?

We can apply the operations that were exemplified as following:

- Translation F1
- OX axis symmetry + translation F2
- OY axis symmetry + translation F3
- Rotation + translation F4
- OX axis symmetry + OY axis symmetry + rotation F5
- Glide reflection +translation F6
- OX axis symmetry + OY axis symmetry + rotation + translation F7

The first type of friezes

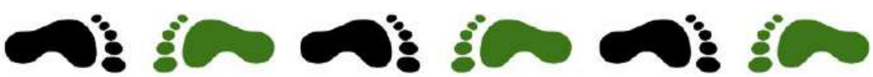
It is the most common and simple type of frieze, and it can be observed in most architectural designs, as it creates the feeling of continuity/infinity. It only contains translations.

Greek temple

The second type of friezes

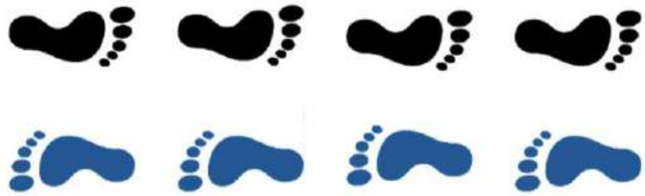
It contains horizontal reflections and translations.

The third type of friezes



It contains vertical reflections and translations.

The fourth type of friezes

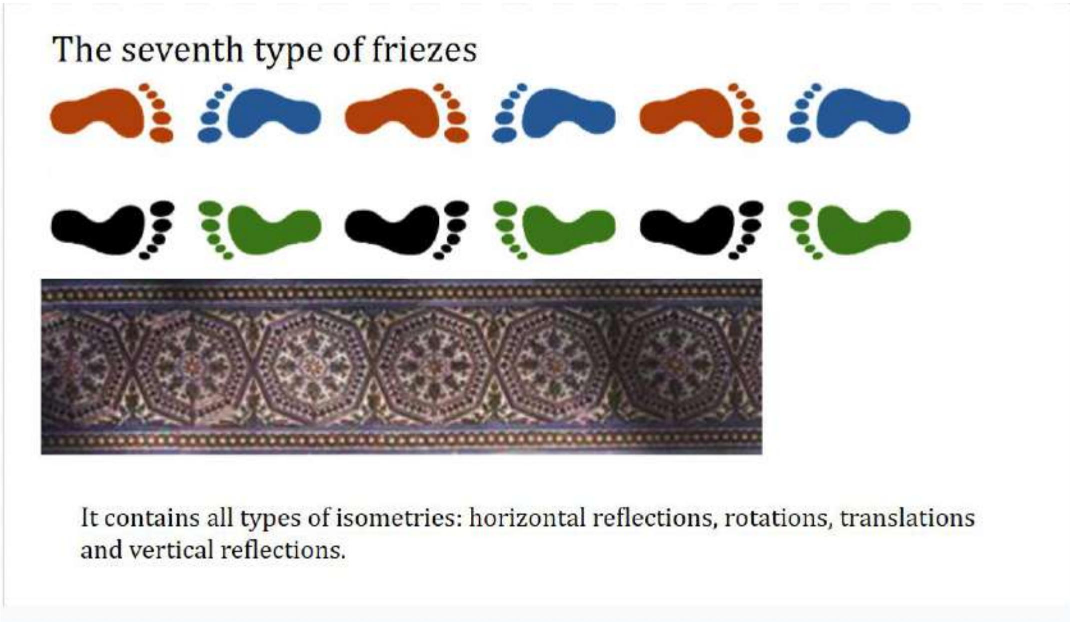


It contains translations and rotations.

Greek temple

The sixth type of friezes

It contains glide reflections and translations.



Conclusion:

- Friezes contain complex and periodic geometric transformations
- There are a total of 4 geometric transformations we identified:translation(T), symmetry(S_{OX},S_{OY}), glide reflection(Gr) and rotation(R)
- Intuitively, there would be 16 possible combinations of these transformations, but only 7 of them are distinct from one another
- We observed these 7 patterns on different friezes around the world