The ping game

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1. Introduction

1.1. The problem

We use a row of counters, white on one side and black on the other.
Starting with a number of n white counters which all need to be turned to black according to the following rule: When we indicate a counter, the counters either side are turned to black.
How should we proceed?

1.2. Results

We have developed a formula that determines the number of moves required in a ping game. Additionally, we have devised a strategy to solve the game irrespective of the initial number of coins flipped to black. Moreover, we've explored similar concepts as those applicable to a single row of coins, but extended to two rows.

2. Results with one row game

2.1 Different way to class the game

The classic game of ping is played on one row. But in order to solve the game more easily we can class the game based on the number of coins. We will separate the game into fours. We will be separating the game into one that has a number of coins written as :

- 4k + 1
- 4k + 2
- 4k + 3
- 4k + 4

with k being the level of the game, with k = 0 as the first level and k = 1 the second one, etc.
This way, the formula and the pattern to solve those coins will be easier to find.

2.2 The formulas and their downsides

There are two formulas. One gives the level of the game based on the number of coins, and the other gives the number of coins you will need to indicate in order to solve the game. The first formula is written as such:

- $\left[\frac{n-1}{4}\right]+1$
- With “n” being the number of coins and “[x]” being the floor of x.
  With this formula we have 1, 2, 3 and four coins being the level one, etc.
  We will not use this formula for a number of coins that can be written as 4k + 1 because games with that number of coins (1 coin, 5 coins, 9 coins, etc) are not solvable, we will explain that later on.

The second formula is written as such:

- **2*l**
  with **l** being the level that we calculated beforehand (the result of $\left[\frac{n-1}{4}\right]+1$).

We can combine those two formulas in order to find the number of times we have to indicate coins in order to solve the game based on the number of coins :

- $2*\left[\frac{n-1}{4}\right]+2$
  which gives us the number of times we need to indicate a coin in order to solve a game based on the number of coins

Unfortunately, the formula cannot be used for any coins that can be written as 4k + 1, it will give an answer that is wrong, for example with 5 coins the formula will say that we need to indicate coins 4 times, but the result is wrong as there is no way to solve a game with 5 coins.

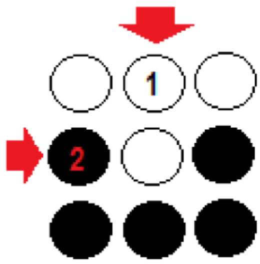
2.3 Result with already flipped coins

If some coins are already turned before the game starts, will we be able to solve it ?
To answer that question, we have to separate the game into two different games, the one with an even number of coins and the one with an odd number of coins.
No matter how many coins have been turned, if you have an even number of total coins there is a way to solve it, it might just be a bit more complicated.
For a game that has an odd number of coins, it is more complicated. If the game has an odd number of odd coins that aren’t already turned (the first coin being 1, the second 2, etc…), then the game is not solvable. Here is why :
The even coins can only be turned by the odd coins and vice versa. But when there is an odd number of coins, you cannot turn only one of them, you have to turn them two by two, because there is no even coin on the corner and the only way to turn only one coin is to indicate a corner.
This is the same reason why 4k + 1 can’t be solved, as when a number of coins is written as such it has an odd number of odd coins and no way to turn only one of them.

3. The different patterns

In order to solve the game, we have decided to create patterns that would solve it in the most efficient way. We figured out that the easiest way to solve an entire row of coins is by dividing it into sets of 3 and 4 coins.

3.1. The set of 3 coins

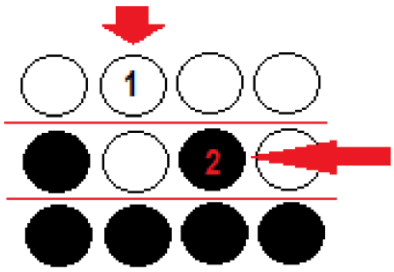


In order to solve such a set, all we have to do is pick the coin in the middle in order to flip the other 2, and then choose the first one to flip the second one (picture 3.1.1).

3.1.1 - Solving the set of 3 coins

3.2 The set of 4 coins

In order to solve this, set we are going to choose the second coin to flip the first and the third, then pick the third coin to flip the second and the fourth coin (picture 3.2.1).

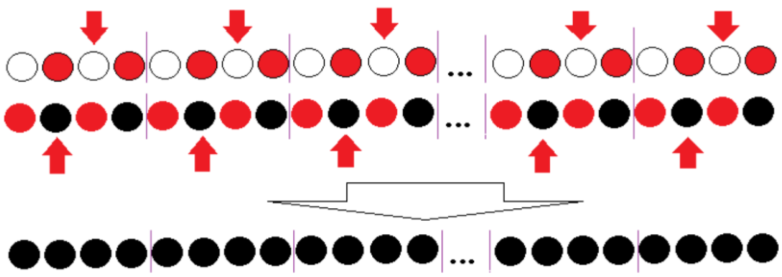


3.2.1 - Solving the set of 4 coins

3.3. 4k coins

Let's say we have 4k coins in total. By dividing the set of coins into groups of 4, we obtain k sets of coins that can be "solved" individually, like described in 3.1.

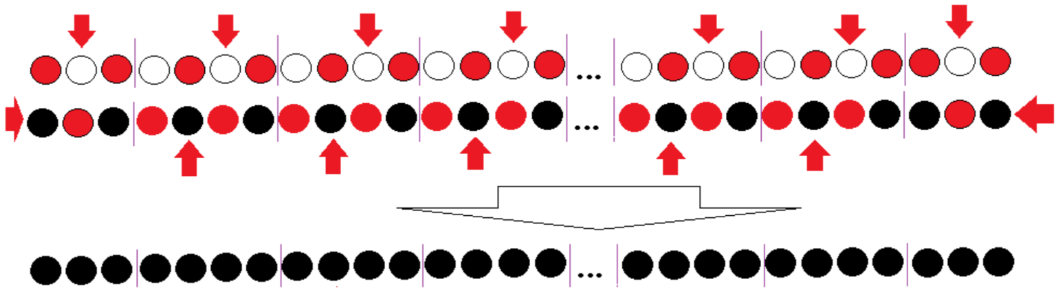
We repeat this process k times, until there are no coins with the white side facing up (picture 3.1.2). Each set of 4 requires 2 steps to be solved, and the operation described previously does not affect any coin from the other groups of 4. Therefore, the minimum number of steps required to solve the problem in this case is 2k.



3.3.1 - Solving the set of 4k coins

3.4. 4k+2 coins

If there are a total of 4k+2 coins, we are going to do something similar, trying to solve sets of 3 and 4 coins. First, we divide the coins into groups: k-1 groups of 4 coins + 2 groups of 3 coins (picture 3.4.1). Now, we will solve each set individually like previously described in 3.1 and 3.2

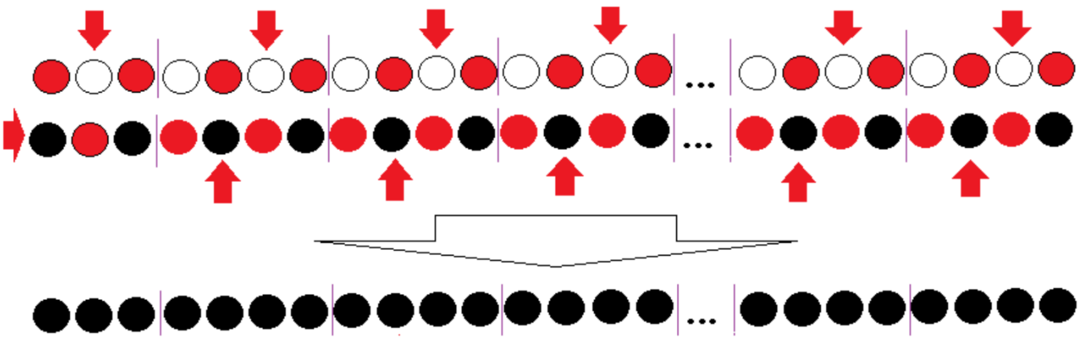


3.4.1 - Solving the set of 4k+2 coins

In order to solve a group of 3 or 4 we need to execute 2 moves for each. Because these moves do not affect other coins from the other sets of 3 or 4, we have k+1 groups of coins in total, each needing 2 moves in order to be solved (k-1 groups of 4 + 2 groups of 3). Thus, to solve the problem in this case, the minimum number of steps we have to execute is 2(k+1).

3.5. 4k+3 coins

This time we split the set of coins into k groups of 4 and 1 set of 3 (picture 3.3.1). We can now "solve" each group of 4 or 3 coins separately, like in the previous case



3.5.1 - Solving the set of 4k+3 coins

The minimum number of steps for solving the problem in this case is still 2(k+1), because we now have k sets of 4 coins and 1 set of 3 coins, each needing exactly 2 moves to be solved.

4. The results with two rows game

4.1. Firsts results

Here we have a game a bit different. This new game has two rows of coins, so the rules aren't exactly the same. When we indicate a coin, the three coins close to it turn in a cross-like shape. The formula we found is approximately the same as the other for the one row game, and give the minimum number of times you indicate a coin you need to end the game:

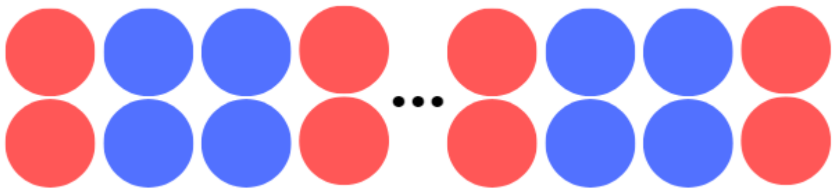
$$2 * \left\lceil \frac{n-1}{6} \right\rceil + 2$$

with n still the number of total coins

Just like the one row game, the formula cannot be used with a number of coins that can be written as 6k + 1.

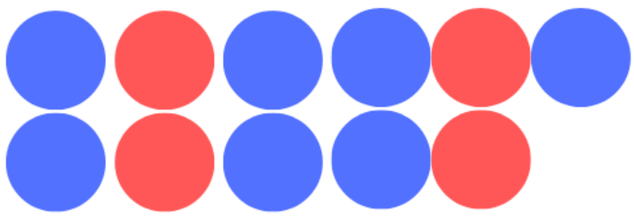
4.2. The patterns

- **with 6k + 2, 6k + 3 and 6k + 4.**
  In order to solve the game as fast as possible with coins written as 6k + 2, 6k + 3, 6k + 4, we have to indicate the first coin starting from the left, for the top and the bottom row, and then we skip two coins and repeat the same process again and again (in the example below the red coins are those that we will need to indicate). The only difference between 6k + 2, 6k + 3, 6k + 4, is that the last column will not look the same, as with 6k + 2, we will have to indicate the last two coins (as seen in the picture below), with 6k + 3 there will be a single coin that won't be indicated (but still turned nonetheless) and for 6k + 4 the very last row will not be indicated at all but still turned.



4.2.1 - Game case 1

- **With 6k + 5 and 6k + 6**
  The same process will be made as the other pattern (6k + 2, etc…), the only difference this time is that we have to start from the second coin from the left, for the top and bottom row, and the skip two coins and repeat. as seen in the example below with the red coins the one that has to be indicated. With 6k + 5 being the example and 6k + 6 having another coin at the bottom row for the last column.



4.2.2 - Game case 2

5. Conclusion

There isn't any formula that works that we found for any number of coins, regardless of the situation when the game is a one row or a two-row game, with $2 * \left[\frac{n-1}{4}\right] + 2$ being the most accurate formula we found for a one row game and $2 * \left[\frac{n-1}{6}\right] + 2$ the closest for a two-row game.

There is still a lot of research that can be made and new ways to improve the formulas and the patterns we found; there might be possible to find a formula that works with any number of rows within a game.