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The Modelling of Fish Stocks

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1 Research Topic

In order to model the management of fish stocks, we consider the deterministic Schaefer model (1954). For that, we take into account a variable M (in tons) which represents the maximum biomass that can live in a certain site and a variable r which represents an intrinsic rate of growth (specific to each species of fish).

If we consider, for $n \geq 1$ the variable X_n , which is the biomass for the year n , the model gives the biomass for the year $n+1$:

$$X_{n+1} = X_n + r \cdot X_n \cdot \left(1 - \frac{X_n}{M}\right) - C,$$

where C is the amount fished by humans.

Let X_0 be the initial biomass (given by a measurement).

Can we model, by modifying the parameters (initial biomass, the rate of reproduction, amount of fish in a year), all the possible situations (extinction, uncontrolled growth of the species)?

2 Useful Tools

For the first approach, we used a C++ algorithm to help get a general idea of the problem. The program below shows the biomass value for all the years until extinction, after reading as input: the maximum biomass (M), the growth rate (r), the fished quantity (C), and the initial biomass (X_0).

```
int M,C;
double r;
double x[1000];

void nextgen( int i )
{
    x[i] = x[i-1] + (r * x[i-1] * (1 - x[i-1]/M)) - C;
}

int main()
{
    cin >> M >> r >> C >> x[0];
    int i=0;
    while( x[i] > 0 )
    {
        i++;
        nextgen(i);
        cout << " " << x[i];
    }
}
```

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```

    cout << i;
    return 0;
}

```

3 Solution

3.1 First Approach

This approach belongs to Diana Harambaş. It was done with the help of Bianca Crişan.

3.1.1 Extinction

The first question we asked ourselves was regarding when will the species go extinct. Hence, we calculated the value of X_n for which X_{n+1} is 0, i.e. we have extinction at the $(n + 1)^{th}$ generation.

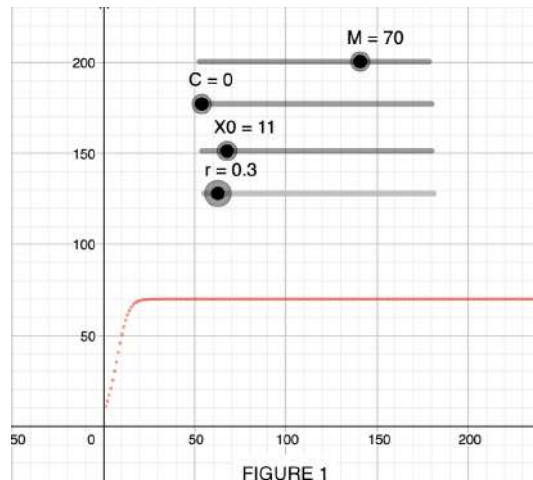
Solving the following second degree equation $(1 + r) \cdot x - \frac{r}{M} \cdot x^2 - C = 0$, we obtain the roots:

$$\frac{(1 + r) + \sqrt{(1 + r)^2 - \frac{4rC}{M}}}{\frac{2r}{M}} \text{ and } \frac{(1 + r) - \sqrt{(1 + r)^2 - \frac{4rC}{M}}}{\frac{2r}{M}}.$$

Obviously, the biomass is positive or null, but never negative. Hence, we need the positive root of the equation. Luckily, both solutions turned out to be positive, as $1 + r > \sqrt{(1 + r)^2 - \frac{4rC}{M}}$. If there is an n for which X_n has one of the values above, then the next year the species will go extinct.

3.1.2 Particular case: $C = 0$

This is the case where no fishing is done and the natural growth of the species is followed. When representing the growth on Geogebra, the graphs converge to M , the maximum biomass. See Figure 1 below:



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Firstly, let us study the variation and possible equilibrium points of this case. Between generation t and $t + 1$ we have a biomass variation of $X_{t+1} - X_t = r \cdot X_t \cdot (1 - \frac{X_t}{M})$. Let such a variation be represented as:

$$\Delta x = \frac{dx}{dt} = r \cdot x \cdot (1 - \frac{x}{M})$$

We can find the equilibrium points by solving $\Delta x = 0$. Thus, 0 and M are the equilibrium points. This means that for any t , if X_t is 0 or M, then $\forall n \geq t, X_n = X_t$.

Knowing that M is the maximum biomass that can be in the lake, we can state that $x \leq M \Rightarrow \frac{x}{M} \leq 1 \Rightarrow \Delta x = r \cdot x \cdot (1 - \frac{x}{M}) \geq 0$. As the variation is always positive, we can affirm that the sequence $(X_n)_{n \geq 1}$ is ascending (or constant). For any n , it is known that $0 \leq X_n \leq M$, which gives the last necessary condition for stating that $(X_n)_{n \geq 1}$ is convergent. Obviously, the sequence must converge to an equilibrium point, so to either M or 0. Given that $(X_n)_{n \geq 1}$ is ascending, we conclude that

$$\lim_{n \rightarrow \infty} X_n = M.$$

So, in a case where no fishing is done, if a fish species starts out with any strictly positive initial biomass, after many generations, the yearly biomass will end up as close to M as possible.

The question one has to ask oneself is: Is this biologically accurate? We have to consider environmental parameters as well, i.e. the lake dimension, the water volume, but most importantly the growth rate of the edible resources. In an ideal case, where the food resources follows a recurrence relation similar to the fish species one and has a greater growth rate, then there will forever be enough food for the fish stock. But, in reality, this might not be the case.

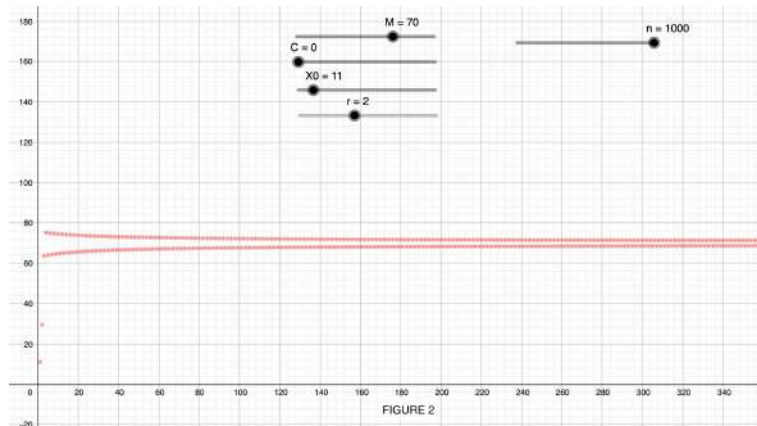
Formula: The differential equation $\frac{dx}{dt} = rx(1 - \frac{x}{M})$ allows us to calculate the value of $X_t, \forall t \geq 1$ by using basic integration rules. Thus,

$$X_t = \frac{M \cdot X_0 \cdot e^{r \cdot t}}{M + X_0 \cdot (e^{r \cdot t} - 1)}, \forall t \geq 1.$$

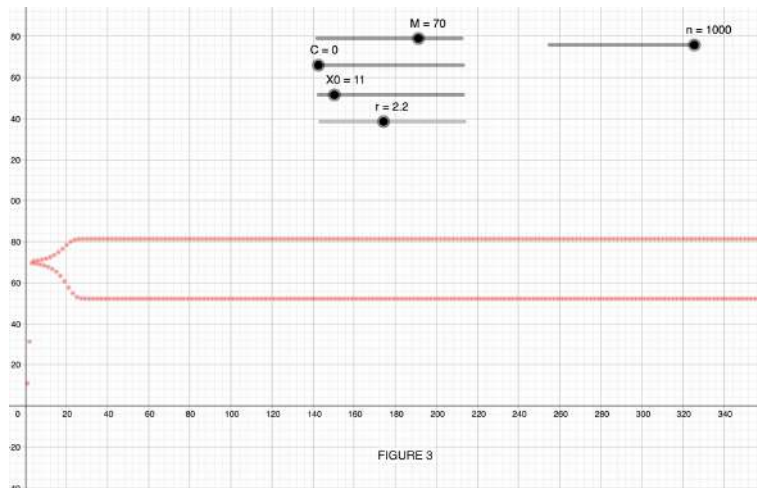
Remarks: We have considered a small rate r that does not let our fish biomass exceed the maximum possible M. But by analysing $x + rx(1 - \frac{x}{M}) > M$, we realised that if there is any $n \geq 1$ for which $r > \frac{M}{X_n}$, then $X_{n+1} > M$, making it impossible for the species to live, and leading to a negative variation and a sudden drop in the biomass of X_{n+2} (the biological species will fight over resources and space).

The graph of such a case can take many forms - forms describing a growth which we considered uncontrollable - but it is easily visible that the values alternate from being greater than M to being less, as shown in Figures 2, 3, 4 and 5. Figure 5 especially is a good example of what we considered as uncontrollable growth.

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Here the rate is 2, the initial biomass is 11 and the maximal one is 70. The complete Geogebra graph shows all values X_n up to $n = 1000$.



In Figure 3, the rate is now 2.2, and the other values are kept the same.

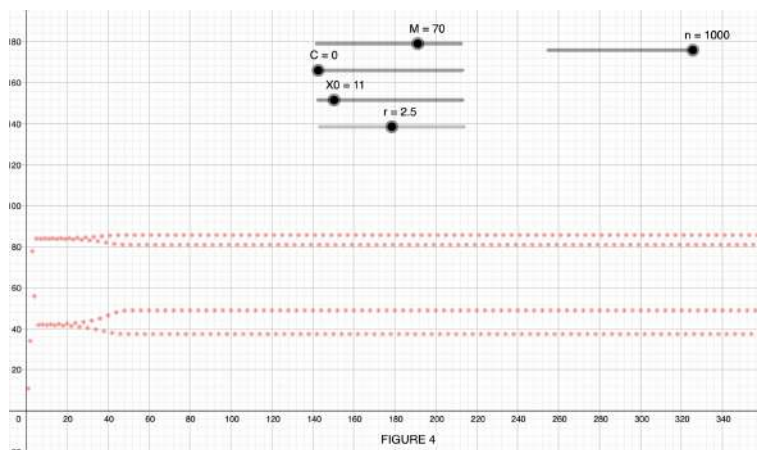
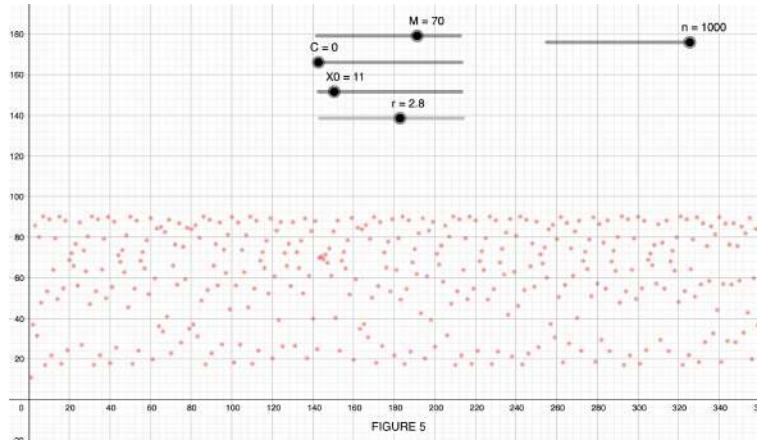


Figure 4 presents the same set of parameters for the initial biomass, the maximal biomass and number of generations. But because we want to emphasize

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what great impact has changing one parameter (namely, the intrinsic growth rate), we set $r = 2.5$.



In Figure 5, the rate is now 2.8, and the other values are the same. It is easily visible that this growth does not seem to follow a pattern, at least not when looking at a small data set of just 1000 generations (on the figure above, just 360 generations were illustrated, to fit the image format and increase visibility on individual values).

3.1.3 General case: *Arbitrary C*

This is the case where the fished amount C is an arbitrary nonzero number. Of course, the inequality $C < M$ is true. From our Geogebra graphs and data tests done with the C++ code, we obtained mostly sequences convergent to 0, so cases where the species will go extinct. This time, the variation differential equation is

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{M}\right) - C,$$

making it harder to integrate if C is arbitrary and not a function of x .

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{M}\right) - C \iff \frac{dx}{rx\left(1 - \frac{x}{M}\right) - C} = dt \iff \frac{dx}{\frac{rx^2}{M} - rx + C} = -dt \iff$$

$$\frac{M}{r} \cdot \frac{dx}{x^2 - Mx + \frac{CM}{r}} = -dt \iff \frac{M}{r} \cdot \frac{dx}{\left(x - \frac{M}{2}\right)^2 - \frac{M^2}{4} + \frac{CM}{r}} = -dt \iff$$

Let $u = x - \frac{M}{2}$. This simplifies our calculations, as now we have to integrate

$$\frac{1}{u^2 + M\left(\frac{C}{r} - \frac{M}{4}\right)}.$$

So, we have this new mathematical relation:

$$\frac{M}{r} \cdot \int \frac{du}{u^2 + M\left(\frac{C}{r} - \frac{M}{4}\right)} = - \int dt \iff \frac{M}{r} \cdot \int \frac{du}{u^2 + M\left(\frac{C}{r} - \frac{M}{4}\right)} = -t + \alpha,$$

where α is a constant (found from the initial conditions). How to solve this depends on the sign of $M\left(\frac{C}{r} - \frac{M}{4}\right)$.

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Case 1: $\frac{C}{r} = \frac{M}{4}$

The new equation is:

$$\frac{M}{r} \cdot \int \frac{du}{u^2} = -t + \alpha_1$$

Basic integration rules lead us to the following equivalent equations:

$$-\frac{M}{r} \cdot \frac{1}{u_t} = -t + \alpha_1 \iff u_t = \frac{M}{(t - \alpha_1)r} \iff X_t = \frac{M}{2} + \frac{M}{(t - \alpha_1)r}$$

The constant should not be independent of X_0 , as it is found by analysing the initial biomass value (generation $t = 0$).

$$X_0 = \frac{M}{2} + \frac{M}{-\alpha_1 r} \iff \frac{X_0 r}{M} - \frac{r}{2} = -\frac{1}{\alpha_1} \iff$$

$$\alpha_1 = \frac{2M}{Mr - 2X_0 r}$$

Thus, the value of the biomass of any t-th generation can be determined by the **formula**:

$$X_t = \frac{M \cdot [t \cdot r \cdot (M - 2X_0) - 4X_0]}{2[t \cdot r \cdot (M - 2X_0) - 2M]}.$$

Case 2: $\frac{C}{r} > \frac{M}{4}$

In this case, we can use another notation to simplify the equation. Hence, let N be a $N = \sqrt{M(\frac{C}{r} - \frac{M}{4})}$. That gives us the equivalent equation:

$$\frac{M}{r} \cdot \int \frac{du}{u^2 + N^2} = -t + \alpha_2.$$

When it comes to solving it, we use the calculus knowledge we have.

$$\frac{M}{r} \frac{1}{N} \cdot \arctan\left(\frac{u_t}{N}\right) = -t + \alpha_2 \iff \frac{u_t}{N} = \tan\left(\frac{Nr(-t + \alpha_2)}{M}\right) \iff$$

$$u_t = \sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)} \cdot \tan\left(\frac{r \cdot \sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)}(-t + \alpha_2)}{M}\right) \iff$$

$$X_t = \frac{M}{2} + \sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)} \cdot \tan\left(\frac{r \cdot \sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)}(-t + \alpha_2)}{M}\right),$$

where α_2 can be found from the initial data, i.e. from X_0 .

$$\alpha_2 = \frac{\sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)} \cdot \arctan\left(\frac{(2X_0 - M) \cdot \sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)}}{2M \cdot \left(\frac{C}{r} - \frac{M}{4}\right)}\right)}{r \cdot \left(\frac{C}{r} - \frac{M}{4}\right)}.$$

Putting together these last two formulas, we obtain that, at the t-th generation, the biomass value will be:

$$X_t = \frac{M}{2} + \sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)} \cdot \tan\left(\frac{r \cdot \sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)}(-t + \frac{\sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)} \cdot \arctan\left(\frac{(2X_0 - M) \cdot \sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)}}{2M \cdot \left(\frac{C}{r} - \frac{M}{4}\right)}\right)}{r \cdot \left(\frac{C}{r} - \frac{M}{4}\right)}}{M}\right).$$

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Case 3: $\frac{C}{r} < \frac{M}{4}$

In this particular case, $M(\frac{C}{r} - \frac{M}{4}) < 0$, making us define the helpful parameter N as $N = \sqrt{M(\frac{M}{4} - \frac{C}{r})}$. The equation now looks like the following:

$$\frac{M}{r} \cdot \int \frac{du}{u^2 - N^2} = -t + \alpha_3.$$

Solving this requires much more advanced knowledge. It is almost basic calculus until we obtain the relation below:

$$\ln \left| \frac{u_t - N}{u_t + N} \right| = \frac{2Nr(-t + \alpha_3)}{M} \iff \left| \frac{u_t - N}{u_t + N} \right| = e^{\frac{2Nr(-t + \alpha_3)}{M}}.$$

This case is what we saw as uncontrollable, because we cannot analyse whether u_t is greater or smaller than N . There might be situations where, for some generations the biomass X_t is smaller than $\frac{M}{2} + N$, and for other generations, greater. We couldn't find a way to find out when such changes will occur, therefore there was no way to determine which is the right formula for this case. We added this on the list of next goals within this research topic.

3.1.4 General case: $C = p \cdot X_n$

This is the case where the fished quantity is calculated every year, as a part p of that generation's biomass, meaning that $C = p \cdot X_n$. Moreover, this is most likely to be the situation closest to reality, as it is the most reasonable when it comes to choosing the fishing biomass with regards to the current quantities.

The recurrence relation here is:

$$X_{n+1} = X_n + r \cdot X_n \cdot \left(1 - \frac{X_n}{M}\right) - p \cdot X_n \iff$$

$$X_{n+1} = X_n + r \cdot X_n \cdot \left(\frac{r-p}{r} - \frac{X_n}{M}\right).$$

The steps we are going to follow are the same as in subsection 3.1.2. Let's study the variation and possible equilibrium points. Defining the variation between generation $t + 1$ and t in the same way as in the previous subsections, we are able to state that $\Delta x = \frac{dx}{dt} = r \cdot x \cdot \left(\frac{r-p}{r} - \frac{x}{M}\right)$.

The equilibrium points are the positive real roots of $\Delta x = 0$. Hence, 0 and $\frac{M \cdot (r-p)}{r}$ are the values we were looking for.

Now, as for the sign of Δx - sign which determines whether the sequence is ascending or descending -, we can easily see that it is all a matter of comparing X_t to $\frac{M \cdot (r-p)}{r}$. While $X_t < \frac{M \cdot (r-p)}{r}$, then the sequence is ascending. Else, while $X_t > \frac{M \cdot (r-p)}{r}$, then the sequence is descending. But because $\frac{M \cdot (r-p)}{r}$ is an equilibrium point and supposing X_0 is non-null, then the sequence should converge to $\frac{M \cdot (r-p)}{r}$.

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We wanted to find the formula for any X_t , by solving the differential equation:

$$\frac{dx}{dt} = r \cdot x \cdot \left(\frac{r-p}{r} - \frac{x}{M} \right).$$

We will arrive to a logarithmic equation involving a modulus and just as we cannot state the monotony of the sequence because we must compare X_t to $\frac{M \cdot (r-p)}{r}$, now we cannot decide the sign of the expression within the modulus.

3.2 Second Approach

This approach belongs to Petru Săveanu and Giret Hugo.

We took the functions g and f where:

$$g(x) = x \cdot (r + 1) - r \cdot \frac{x^2}{M} - C \quad \text{and} \quad f(x) = r \cdot x - r \cdot x \cdot \frac{x}{M} - C$$

$$g(x) = x + f(x)$$

In other words, the function f is the difference between 2 years and

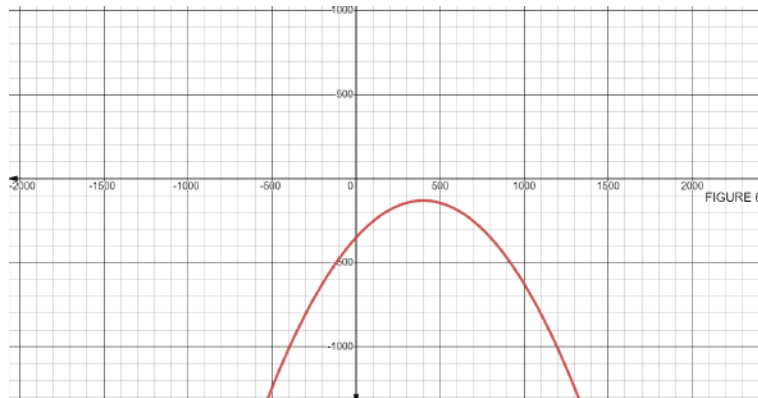
$$\Delta = r^2 - 4 \cdot r \cdot \frac{C}{M}$$

We chose to study the function f because it explains the evolution from year to year. For example, if the function is positive, it means an increase in quantity, if it is negative it will lead to extinction, and if it is once zero, the result will be stagnation.

3.2.1 Case 1: $\Delta < 0$

$$\forall x, f(x) < 0 \quad \text{so,} \quad g(x) = x + f(x) < x$$

Then $X_0 > X_1 > \dots > X_n$



Considering the graph of the function f, the differences will get bigger and bigger, because by approaching the quantity of 0, the origin of the system, the graph tends to minus infinity, so the result will be obvious extinction.

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3.2.2 Case 2: $\Delta = 0$

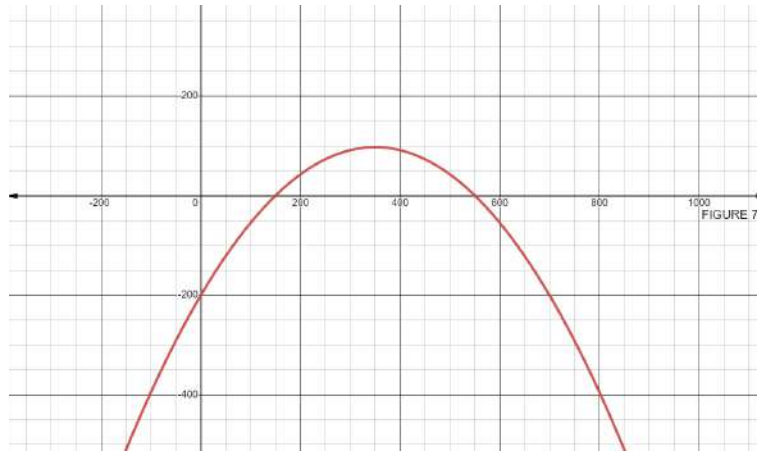
$$\frac{C}{r} = \frac{M}{4} \Rightarrow f\left(\frac{M}{2}\right) = r \cdot \frac{M}{2} \cdot \left(1 - \frac{\frac{M}{2}}{M}\right) - C = \frac{r \cdot M}{4} - C = 0$$

The only point where $f(x) = 0$ is $\frac{M}{2}$. If X_0 is less than $\frac{M}{2}$ the result will be extinction, because we are in the same situation as in the previous case, but we try to find the balance. Thus, we will look for the points that bring the biomass below the value of $\frac{M}{2}$, trying to calculate the distance from $\frac{M}{2}$, called "a".

$$g\left(\frac{M}{2} + a\right) < \frac{M}{2} \iff \frac{M}{2} + a + \left(\frac{M}{2} + a\right) \cdot r \cdot \left(1 - \frac{\frac{M}{2} + a}{M}\right) < \frac{M}{2} \iff a < a^2 \cdot \frac{M}{r} \iff \frac{M}{r} < a$$

This means that if X_0 belongs to the interval $\left(\frac{M}{2}, \frac{M}{2} + \frac{M}{r}\right)$, $g(x) > \frac{M}{2}$, but because $f(x) \leq 0$, $X_0 < X_1 < \dots$, the decrease limit being $\frac{M}{2}$. So $\forall x$, if X_0 does not belong to that range, the result will be extinction.

3.2.3 Case 3: $0 < \Delta \leq 1$



We chose this case because we can choose a clearly defined interval to obtain balance, and uncontrolled growth is impossible. We used the graph of the function f and the first derivative of the function g to obtain intervals of monotony. $g'(x) = r + 1 - x \cdot \frac{2 \cdot r}{M}$ For any x in range of x_1 and x_2 , where $f(x_1) = 0$ and $f(x_2) = 0$, $g'(x) \geq 0$

$$g'(x_2) = r + 1 - \left(\frac{r + \sqrt{\Delta}}{\frac{2 \cdot r}{M}}\right) \cdot \frac{2 \cdot r}{M} = r + 1 - r - \sqrt{\Delta} = 1 - \sqrt{\Delta}$$

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$$x_2 > x_1 \Rightarrow g'(x_1) > g'(x_2) = 1 - \sqrt{\Delta} \geq 1 - 1 = 0$$

$$\Rightarrow x_1 = g(x_1) < g(x) < g(x_2) = x_2 \text{ if } x_1 < x < x_2 .$$

Using $f(x) > 0$ where $x_1 < x < x_2$ we obtain: $x_1 < X_0 < X_1 < X_2 < \dots < X_n < x_2$, balance, if $x_1 < X_0 < x_2$ and $x_1 > X_0 > X_1 > X_2 > \dots > X_n$ meaning extinction if $X_0 < x_1$. As in the previous case, we have a limit, this being x_2 .

For $X_0 > x_2$ we can obtain balance or extinction, depending on the distance from x_2 .

The abscissa of the middle of the of $g(x)$: $m = \frac{M(r+1)}{2r}$. So by using symmetry we obtain an interval of balance: $x_1 < X_0 < 2m - x_1$.

3.2.4 Case 4: $\Delta > 1$

This is the only situation when X_n could be greater than M . Due to the very large variation of values in two consecutive years, we studied the situation when X_1 is greater than M .

We took the function h , where $h(x) = g(x) - M$ and we calculated the roots of the equation.

$$h(x) = g(x) - M = x \cdot (r + 1) - r \cdot \frac{x}{M} - C - M \Rightarrow \Delta' = (r + 1)^2 - 4 \cdot r \cdot \frac{C + M}{M}$$

Then we consider the points $P = \frac{r+1-\sqrt{\Delta'}}{2 \cdot \frac{r}{M}}$ and $W = \frac{r+1+\sqrt{\Delta'}}{2 \cdot \frac{r}{M}}$ for which $g(P) = M$ and $g(W) = M$.

Because X_0 is smaller than M , and $g(x) > x$ only if $x_1 < x < x_2$, we will create an interval by joining the interval (x_1, x_2) with the interval (P, W) . If X_0 belongs to that interval, $X_1 \geq M$.

Using the 4 cases we can get all three possible situations.

3.2.5 Maximal fishing

The maximal fishing is the fishing value where we fish as much as fish we can without causing the extinction of the specie.

For $X_n < \frac{M}{2}$ and $X_n > \frac{M}{2} + \frac{M}{r}$:

$$C_{max} = \frac{-M \cdot \left(\frac{2rX_n}{M} - \frac{2}{r}\right)^2 - r^2}{4r}$$

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As we have previously demonstrated, for $X_n > \frac{M}{2}$ and $X_n < \frac{M}{2} + \frac{M}{r}$, the maximal fishing is obtained when $\Delta = 0$ and equals to :

$$C_{max} = \frac{rM}{4}$$

4 Conclusions

Looking at the first approach, we can observe that formulas can be found, but they are different and depend on the relationship between the given parameters C, M and r. Moreover, some cases are harder to follow and, from the knowledge we possess now, can be considered as cases of uncontrollable growth.

From our previous work, we observed different types of curve depending on the $\Delta = r^2 - 4 \cdot r \cdot \frac{C}{M}$ value: The extinction curve for $\Delta < 0$, the constant curve, the "alternating"/"sinusoidal" curve, the scattered one, etc.

By introducing a new environmental variable V_n (which can only take the values 0.5, 1, and 1.5) to this extent: $X_{n+1} = (X_n + r \cdot X_n \cdot (1 - \frac{X_n}{M}) - C) \cdot V_n$, we obtain unusual cases, creating new types of curves that seems non-periodic and hard to define. Studying this evolution type is one of our next research goals.

The Schaefer Model is not a perfect one: it is a discrete mathematical model, meaning that the fish population is calculated every year; in real life, environmental facts and events could influence so much that the fish could disappear in 6 months, for example, and so the model would not work. To be more accurate, we should take more parameters (related to the aquatic habitat) into consideration.

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5 Other contributors

Other contributors to this research topic within the "Mathematics for Sustainable Development" project were Jucan Codruța, Cătărig Horațiu, Coroiu Alexandru, Șipos Sony, Czornomaz Dimitri, Dekeyser Alexis, Brittain Théo, Mekarcha Diégo, Le Bris Lou-Anne, Fernandes Lucas, Virga Valentin.

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