REPEATED PATTERNS

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1. Research topic

We apply a ruler to a black triangle whose point is up. We connect the mid-point of its sides and then remove the newly formed triangle, thus obtaining three new black triangles. We observe one triangle with side 1. What can be said about the number of triangles and points of x steps of the above rule?

2. How we solved the problem

We started by drawing the triangle in each phase. In phase 1, we got a complete, equilateral triangle. Then, to obtain the figure representative for phase 2, we connected the mid-points of each side of the initial triangle, and we formed an upside-down triangle. We removed the upside-down triangle because we considered it as a hole. By doing this, we obtained 3 triangles in phase 2. Going forward, for phase three, we took the figure that we obtained in the second phase, and we applied the same rule, but now for each of the 3 newly obtained right-side-up triangles. So, we connected the midpoints of each side of one triangle to obtain an upside-down triangle, which we removed afterwards. We repeated this procedure three times, for each triangle. Finally, we got 9 triangles in phase 3. For phase 4, we did the exact same thing but repeated 9 times. As if in each newly formed triangle, we formed 3 more right-side-up triangles and one upside-down triangle that we removed each time. This pattern goes on and on to the infinite and the more the phase grows, you get more and more small triangles.

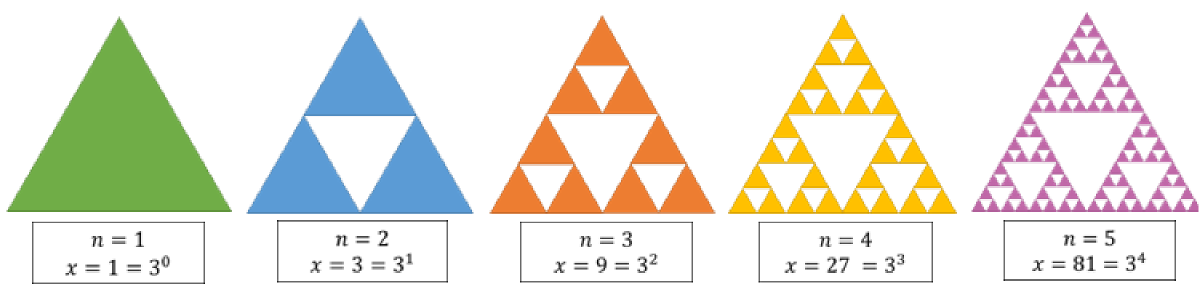

Figure 1. The first five phases of the Sierpiński triangle and the corresponding formulas to each phase

For it to be easier to study the formulas, we used some symbols, which are explained in the following legend:

- x = number of triangles
- n = phase/step

To find the formulas, we drew the geometric figures for the first 5 phases (see *Figure 1*) and then we counted the number of triangles we got in each phase. So, in the first phase, for n=1, we got an x=1 triangle. In the second phase, for n=2, we obtained x=3 triangles. For n=3, we got x=9 triangles, for n=4, we have x=27 triangles and for n=5, we got x=81 triangles. Just by looking at the numbers that represent the number of triangles in each phase, we observed that they are all powers of 3. For example, $1=3^0$, $3=3^1$, $9=3^2$, $27=3^3$, $81=3^4$. In order to find a formula which connects the number of triangles to the phase we are in, we studied the way in which the power of three modifies. We noticed that the exponent of the power of three is $n-1$, where n is the phase number.

So, the formula that expresses the number of triangles according to the phase is:

$$\boxed{\mathbf{x = 3^{n-1}}}$$

To prove that the formula works no matter how big the phase gets, that it is universally valid, we use mathematical induction:

$$(1): 1 = 3^{1-1} \Leftrightarrow 1 = 3^0 \ \ (T)$$
$$\mathrm{p}\ (2): 3 = 3^{2-1} \Leftrightarrow 3 = 3^1 \ \ (T)$$
$$\mathrm{p(k)}: 3^{\mathrm{k}-1}\ (T)$$
$$\mathrm{p}\ (k)\ \ (T) \rightarrow p(k+1)\ \ (\mathrm{T})$$
$$\mathrm{p}\ (k+1): 3^{(k+1)-1} = 3^k,$$
$$\text{But we have the relation :}$$
$$p(2) \cdot p(k) \quad (T) \Rightarrow 3^1 \cdot 3^{k-1}$$
$$= 3^{k-1+1} = 3^k = p(k+1)$$
$$\Rightarrow p(k+1)(T)$$

3. Other formulas

We found a relation between the number of triangles that we obtained (x) and the phase we are in (n). But we could not stop here. We explored more the properties that this special figure has. We went on by studying the number of angles, the total perimeter and the total surface. And we found out even more formulas and interesting facts about this triangle that goes on forever.

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We drew again the triangles for the first 5 phases (see *Figure 2*), but now we looked into other things, so we introduce a few more new symbols that are explained in the following legend:

- n = phase/ step
- a= number of angles
- Pt = total perimeter
- St = total surface
- l = side of the initial triangle (in phase 1)

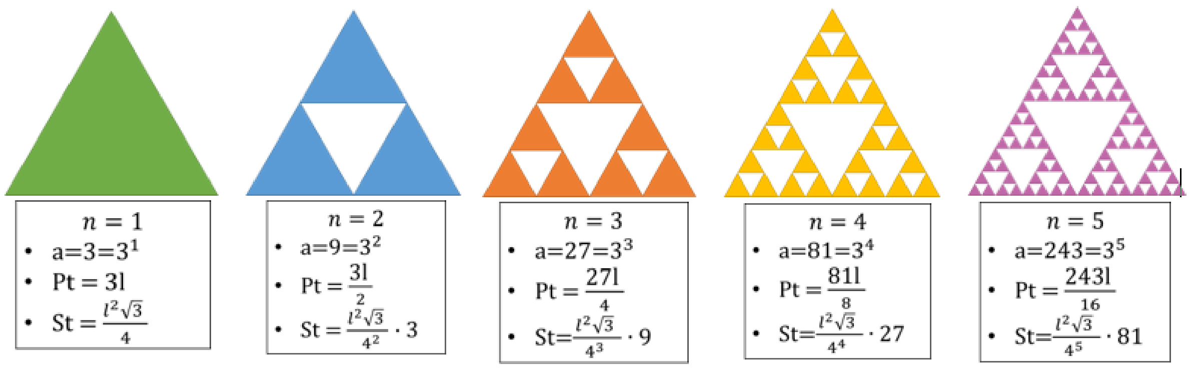

Figure 2. The first five phases of the Sierpiński triangle and the other corresponding formulas to each phase

2.1 The number of angles

To find out a formula for the number of angles we used the same principle as we did to find the first formula, the one with the number of triangles. So, we know that all the triangles we obtain no matter which phase we are in are equilateral. One of their most important properties is that they have all 3 angles equal. By counting the number of angles in each phase, we got the following values: in the first phase we have, of course, a = 3 angles; in the second phase we have 3 similar triangles which have each 3 angles, so we got a = 9 angles; in phase 3 we have 9 triangles which have each 3 angles, so we obtain a=27 angles, in phase 4, a = 81 angles and in phase 5, a = 243 angles. We can easily observe a rule, which is that the number of angles has the form of 3 at the power of the phase we are in (n). The general formula which calculates the number of angles according to the phase we are in is: $\boxed{a = 3^n}$

To prove that this formula is correct and that it is universally valid we used again mathematical induction:

p(1): 3^1=3 (T)
p(2): 3^2=9 (T)
p(k) (T) → p(k+1) (T)
p($k+1$): 3^{k+1}=$3^k \cdot 1$
But we have the relation:
$p(1) \cdot p(k) \quad (T)$
$\Rightarrow 3^1 \cdot 3^k = 3^{k+1} = 3^{k+1}$
$= p(k+1)$
$\Rightarrow p(k+1)(T)$

2.2 The total perimeter and the total surface

As we said before, we considered the initial triangle (the triangle in phase 1) an equilateral triangle. So, we know that all its sides have the same length. We have some special formulas for the perimeter and the area of an equilateral triangle, which are: $\boxed{P=3l}$ and $\boxed{S=\frac{l^2\sqrt{3}}{4}}$

We used these two particular formulas to determine the total surface and the total perimeter for each figure in each phase. The principle of the formulas stays the same, the only thing that modifies is the length of the sides of the triangle (l) as we divide the initial triangle into smaller and smaller triangles each time we get to a new phase. So, the side varies, in the same way, the triangle gets divided: for phase 2, we have the side $\frac{l}{2}$, for phase 3, we got the side $\frac{l}{2^2}$, for phase 4 we have the side $\frac{l}{2^3}$ and for phase 5, we have the side $\frac{l}{2^4}$. As we can see, the side gets divided by powers of 2. So, to obtain the total perimeter and the total surface in a specific phase, we just replace the constant which represents the length of the triangle's side with its value according to how much it gets divided in that phase. Using this method, we got the following formulas written in a general form, so they work in every phase:

$$\boxed{P_t = \frac{3^n \cdot l}{2^{n-1}}} \qquad \boxed{S_t = \frac{l^2\sqrt{3}}{4^n} \cdot 3^{n-1}}$$

4. The most interesting facts

By studying the way in which the total surface and the total perimeter vary, we noticed some really interesting and unusual properties that make this particular geometrical figure so special and unique:

- ∆ The geometric figure that we obtained (is a figure that has a finite surface and an infinite perimeter). This means that the bigger the phase gets, the perimeter grows and the surface decreases as we get more small triangles, which are all included in the same big triangle that we have for phase 1. Just like this, the perimeter of this special figure tends to the infinit while the surface tends to 0.

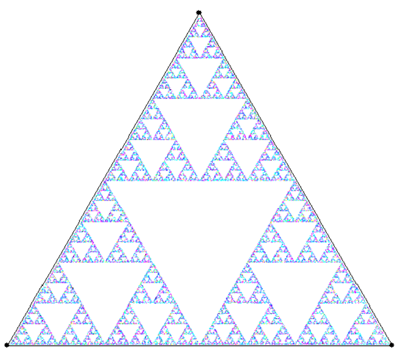

Figure 3 . The Sierpiński triangle

- ∆ Theoretically, this drawing goes on forever. You can divide each triangle into the infinite. But reaching the infinite is not a possible thing in our real life, at least not for us, humans... So, the question we should all ask ourselves is how far can this "magical triangle" really go?

- ∆ Of course, how much you can divide the figure depends on how big you consider the length of the initial triangle's side. The bigger the length is, the more you can divide the triangle. But even if you take a triangle with a side 5 meters or more, and you keep dividing that into smaller and smaller triangles, the 500 meters will turn into 250, then into 125 and just like that, you will find yourself at a point in which theoretically you can still divide even the smallest triangle, but practically, your pencil can't possibly draw that. So is the end of this triangle the atom, the smallest particle that can no longer be divided? And does it stop at some point, or does it go to the infinite?

5. 3D pyramid

We also created a model of a 3D pyramid (see *Figure 4*) in Thinke3rcad which follows the same pattern and we printed it with the 3D printer. By doing this, we transposed the drawing corresponding to the triangle in the third phase into a three-dimensional geometric figure.

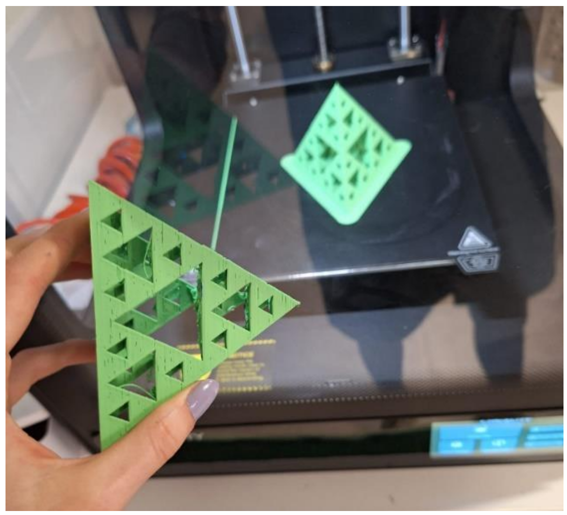

Figure 4. A model of a 3D pyramid

6. Schoolyard drawing

Many people perceive mathematics as a theoretical and complex discipline, missing the enjoyable part of it. Through the use of creative projects, this misconception regarding the viability of this subject can be changed. Therefore, we had the idea of bringing our research topic to “life” by drawing a massive equilateral triangle (the length of the sides being 5.12 meters) with chalk in our schoolyard (see *Figure 5*).

To do this precisely we had the idea to make cardboard templates of the triangles in various dimensions, which represented the phase of each triangle. The smallest one had a side of 16 cm and we had a total of 6 phases.

A particularly valuable feature of our project was involving our younger schoolmates, each bringing extra energy. With their help, what was estimated to take us up to 4 hours to complete was done successfully in only 2 hours. Therefore, mathematics succeeded in bringing us all together.

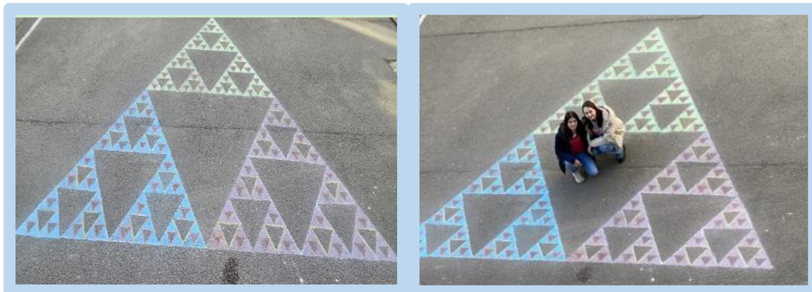

Figure 5. The drawing with chalk in our schoolyard

7. The Sierpinski Triangle

The Sierpiński triangle *(see Figure 3)*, also known as the Sierpiński gasket or the Sierpiński sieve. This geometric object preserves the general geometry of an equilateral triangle and is an important example of a fractal, as it is one of the first ones ever defined. It is made by repeatedly splitting the triangle into smaller equilateral triangles in a recursive manner that goes on forever.

The Sierpiński triangle was first designed as a particular kind of curve, but it has subsequently evolved into one of the most basic illustrations of self-similar sets in mathematics. Because of its self-similarity, the Sierpiński triangle pattern can be reproduced at any size and still maintain its structure and characteristics.

The origin of this construction belongs to Wacław Franciszek Sierpiński, a Polish mathematician who discovered it in 1916 while investigating sets that are not measurable in the usual sense.

8. Fractals

The Sierpiński triangle is used as an introduction to them when beginning the study of fractals and sets. It demonstrated the recursive properties of self-similar fractals and is a good starting place to gain insight. Fractals are never-ending patterns that repeat themselves at different scales. Fractals occur in all sorts of places in nature: in lightning bolts, clouds, frost crystals, blood vessels *(see Figure 7)*, seashells *(see Figure 8)*, hurricanes *(see Figure 6)* and river networks.



Figure 6. Hurricanes *Figure 7. Blood vessels* *Figure. 8. Seashells*

Most of us unknowingly use this triangular fractal every day:

1. Antennas based on this particular pattern can be found in most mobile phones.
2. Fractal medicine. Knowledge of fractals is especially useful in medical diagnoses, including for cancer. Since healthy human blood vessel cells typically grow in an orderly fractal pattern, cancerous cells, which grow in an abnormal fashion, become easier to detect. This form of fractal analysis makes distinguishing between healthy cells and signs of concern much easier.
3. Fractal cities. Some cities tend to grow in fractal patterns over time and are called fractal cities. As a large fractal city absorbs its neighbouring towns and villages, the pattern developed resembles a self-similar structure that seems random at first but is a dynamic network that may prove to be more efficient than modern "pre-planned" cities.

9. Conclusion

Through this project, "Repeated patterns", we had the opportunity to expand our knowledge about mathematical concepts and the world that surrounds us. It was not a very difficult topic, but it allowed us to approach it from several perspectives. As a result, we transformed this project into a complex one while ensuring everyone could understand it.

References

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