A square in a curve

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1. Introduction

The problem

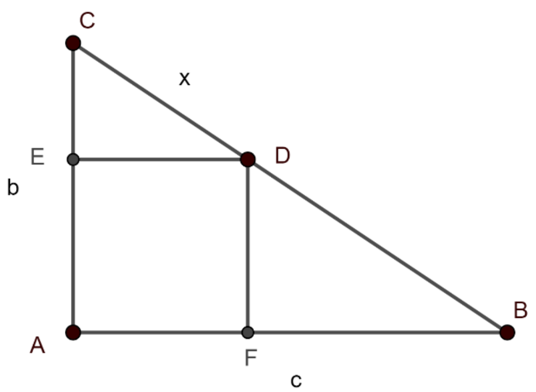
We draw a closed curve on the plane. Can we always choose 4 distinct points on this curve, A, B, C, D, such that ABCD forms a square?

Results

We took some particular curves and we found out that there exist four points on these particular curves to obtained a square.

2. Right-Angled Triangle

Position 1



Let D ∈ BC, CD = x and x < $\sqrt{b^2 + c^2}$ (lenght of BC)

DE ∥ AB, E ∈ AC, DF ∥ AC, F ∈ AB, so AFDE is a rectangle.

$\triangle CED \sim \triangle CAB \Rightarrow \frac{ED}{c} = \frac{x}{\sqrt{b^2+c^2}} \Rightarrow \mathbf{ED} = \frac{cx}{\sqrt{b^2+c^2}}$

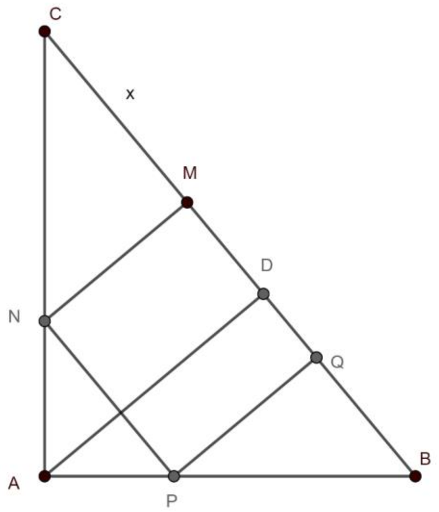
$\triangle BDF \sim \triangle BCA \;\; \Rightarrow \frac{DF}{b} = \frac{\sqrt{b+c^2}-x}{\sqrt{b^2+c^2}} \;\; \Rightarrow$ DF=$\frac{b\left(\sqrt{b^2+c^2}-x\right)}{\sqrt{b^2+c^2}}$

AFDE- square ⇔ ED=DF ⇔ $\frac{cx}{\sqrt{b^2+c^2}} = \frac{b\left(\sqrt{b^2+c^2}-x\right)}{\sqrt{b^2+c^2}} \Leftrightarrow cx = b\sqrt{b^2+c^2} - bx \Leftrightarrow (b+c)x = b\sqrt{b^2+c^2}$

⇔ $0 < x = \frac{b\sqrt{b+c^2}}{b+c} < \sqrt{b^2+c^2}$

The side of the square is DE = $\frac{c \cdot \frac{b\sqrt{b+c^2}}{b+c}}{\sqrt{b^2+c^2}} \Rightarrow$ DE = $\frac{bc}{b+c}$, so in this case the square exists inside the right-agled triangle.

Position 2



Let point M on BC. $M \neq D$, D is the projection of point A onto the line BC.

CM=x, x< $\sqrt{c^2+b^2}$ and x<CD

MN⊥BC, $N \in AC$, NP ∥BC, $P \in AB$, PQ⊥BC, $Q \in BC$ so MNPQ is a rectangle.
MNQP square ⇔ MN=NP.

$\triangle CMN \sim \triangle CAB \Rightarrow \frac{CM}{CA} = \frac{MN}{AB} = \frac{CN}{BC} \Rightarrow \;\; \frac{x}{b} = \frac{MN}{c} = \frac{CN}{\sqrt{c^2+b^2}} \Rightarrow$ CN=$\frac{x\sqrt{c^2+b^2}}{b}$

$\triangle ANP \sim \triangle ACB \Rightarrow \frac{AN}{AC} = \frac{NP}{BC} = \frac{AP}{AB} \Rightarrow \;\; \frac{AN}{b} = \frac{NP}{\sqrt{c^2+b^2}} = \frac{AP}{c} \Rightarrow$ NP=$\frac{AN\sqrt{c^2+b^2}}{b}$,

But AN=$\mathbf{AC} - \mathbf{NC} = \mathbf{b} - \frac{x\sqrt{c^2+b^2}}{b}$.

MN = $\frac{cx}{b}$ and NP = $\frac{AN\sqrt{c^2+b^2}}{b} = \frac{\left(b - \frac{x\sqrt{c^2+b^2}}{b}\right)\sqrt{c^2+b^2}}{b}$

MN=NP $\Leftrightarrow \frac{cx}{b} = \frac{\left(b-\frac{x\sqrt{c^2+b^2}}{b}\right)\sqrt{c^2+b^2}}{b} \Leftrightarrow cx = b\sqrt{c^2+b^2} - \frac{x(b^2+c^2)}{b}$

$\Leftrightarrow$ x=$\frac{b^2\sqrt{b^2+c^2}}{b^2+bc+c^2}$, x<CD $\Rightarrow$ The square exists inside the right-agled triangle in with this position,

The side of the square is MN=$\frac{bc\sqrt{b^2+c^2}}{b^2+bc+c^2}$.

Which area is bigger?

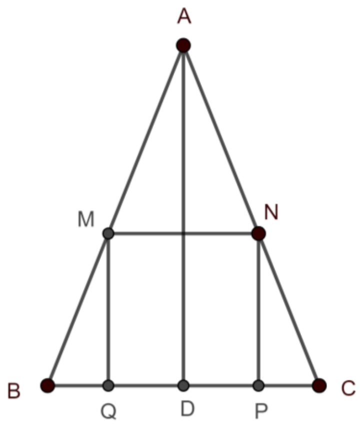
After we consider those two positions we wanted to find which area is bigger, in position 1 or in position 2?

We have to compare $\frac{bc}{b+c}$ with $\frac{bc\sqrt{b^2+c^2}}{b^2+bc+c^2}$

$\frac{bc\sqrt{b^2+c^2}}{b^2+bc+c^2} < \frac{bc}{b+c} \Leftrightarrow (b+c)\sqrt{b^2+c^2} < b^2+bc+c^2 \Leftrightarrow (b^2+2bc+c^2)(b^2+c^2) < b^4+b^2c^2+c^4+2b^3c+2bc^3+2b^2c^2 \Leftrightarrow 0 < b^2c^2$, true.

So, the square inscribed into the right-angled triangle in position 1 is bigger than the square in position 2.

3. The isosceles triangle



BC=2a; AB=AC=b; Let be N on AC, AN=x =>NC=b-x.

NP⊥BC, $P \in BC$, MN ll BC, $N \in AB$, MQ⊥BC, $Q \in BC$.

AD=$\sqrt{b^2-a^2}$ (Theorem of Pythagoras in ΔADC)

ΔAMN~ΔABC=> $\frac{MN}{2a} = \frac{x}{b}$=> MN = $\frac{2ax}{b}$

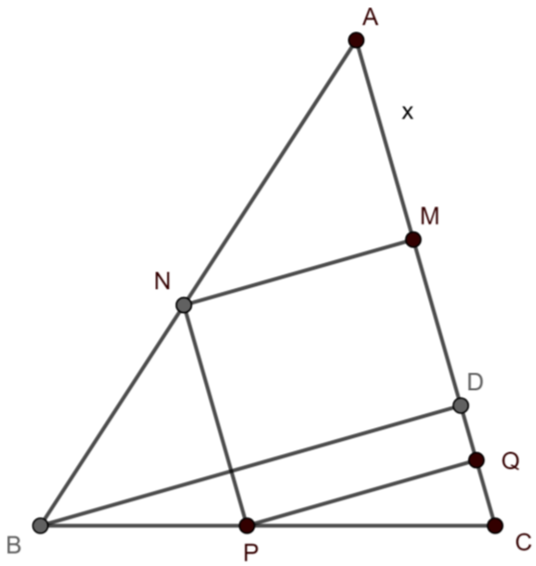
ΔCPN~ΔCDA=> $\frac{NP}{\sqrt{b^2-a^2}} = \frac{b-x}{b}$. MNPQ square MN=NP$\Leftrightarrow$

$\Leftrightarrow$ 2ax=b $\cdot \sqrt{b^2-a^2}$- x $\cdot \sqrt{b^2-a^2} \Leftrightarrow$ x(2a + $\sqrt{b^2-a^2}$) = b $\cdot \sqrt{b^2-a^2}$

$\Leftrightarrow$ x = $\frac{b \cdot \sqrt{b^2-a^2}}{2a+\sqrt{b^2-a^2}}$ < b. The square exists inside the isosceles triangle in this position.

The square side is MN = $\frac{2a}{b} \cdot \frac{b\sqrt{b^2-a^2}}{2a+\sqrt{b^2-a^2}}$, MN = $\frac{2a\sqrt{b^2-a^2}}{2a+\sqrt{b^2-a^2}}$.

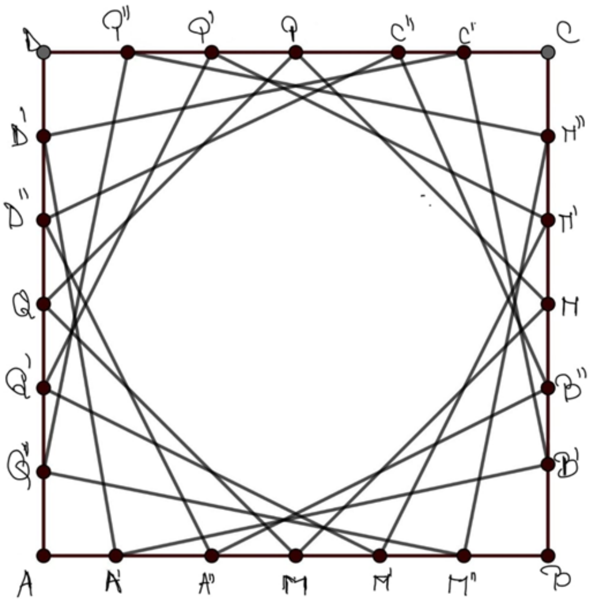
4. The Scalen triangle



Let M be a point on AC, AM=x, x<AD (D is the projection of point B onto the line AC), MN⊥AC, $N \in AB$
NP ll AC, $P \in BC$, PQ⊥AC, $Q \in AC$. MNPQ is a rectangle.
AB = c; BC = a; AC = b
ΔAMN~ΔADB => $\frac{x}{AD} = \frac{MN}{BD} = \frac{AN}{c}$
ΔBNP~ΔBAC => $\frac{BN}{c} = \frac{NP}{b} \Rightarrow$ NP = $\frac{BN \cdot b}{c}$
BN = AB - AN = c - $\frac{cx}{AD}$
MN = NP ⇔ $\frac{x \cdot BD}{AD} = \frac{(c - \frac{cx}{AD}) \cdot b}{c} \Leftrightarrow \frac{x \cdot BD}{AD} = \frac{(AD - x) \cdot b}{AD} \Leftrightarrow x \cdot \boldsymbol{BD} = AD \cdot \boldsymbol{b} - x \cdot \boldsymbol{b}$

x(BD + b) = AD $\cdot\, \boldsymbol{b} \Leftrightarrow x = \frac{b \cdot AD}{BD + b}$ < AD. . The square exists always inside the scalen triangles.

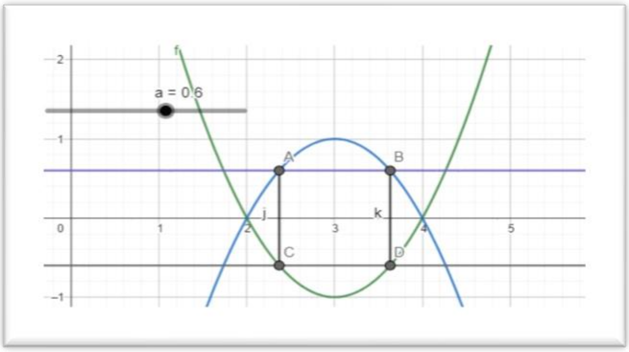
5. Squares in a square



ABCD is a square. We take 5 point on each side AA'= A'A''=...=$\frac{AB}{6}$ and similar for the rest of the points.
ΔA'AD',ΔD'DC',ΔC'CB',ΔB'BA' are right angle triangles with BA'=AD'=DC'=CB' and AA'=DD'=CC'=BB'.
ΔA'AD'≡ΔD'DC'≡ΔC'CB'≡ΔB'BA'=>A'B'=B'C'=C'D'=A'D'=> A'B'C'D' rhombus.
ΔB'BA'-right-angled triangle=>∠BA'B'=y, ∠A'B'B=x, x + y = 90°
ΔA'AD'- right-angled triangle=>∠AA'D=x, ∠AD'A'= y
∠AA'D'+ ∠BA'B'= x + y =90° and D'A'B'= 180°- (∠AA'D'+∠BA'B') =90°, so A'B'C'D' is square.
Analog for M'N'P'Q', A''B''C''D' and all the rest of similar them are squares.
The result is the same if we consider four points A',B',C,'D' on each side of the square with the condition AA'= BB'= CC'= DD'. Using the same method, we can also deduce ∠OBD=∠OBF, obtain congruent triangles and finally A'B'C'D' is square.

6. Square in the curve bounded by two parabolas

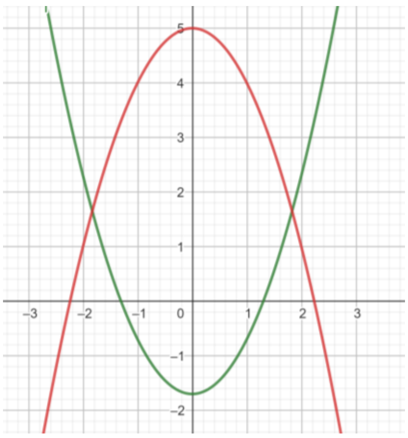
Particular case 1



We consider $f: \mathbb{R} \to \mathbb{R}$, f(x) = x^2-6x +8, $g: \mathbb{R} \to \mathbb{R}$ g(x) = $-x^2$+6x-8 = -f(x).
f(x)=g(x) ⇔ f(x)=-f(x) ⇔ 2f(x)=0 ⇔f(x)=0. We'll have the points with the coordinates (2,0) and (4,0).
We will determine the coordinates of A(x, f(x)) so that ABDC is a square A and B are both symmetrical to the top of the concave parabola V(3,g(3)) ⇔V(3,1).
Let A (3-x,g(3-x)); B(3+x),g(3 +x)), C(3-x,f(3-x)), D(3+x,f(3 +x)) x>0
ABDC square ⇔2x =2g(3+x) ⇔x= - $(3+x)^2$+ 6(3+x) - 8⇔ x^2+ x-1=0, $x_{1,2}=\frac{-1\pm\sqrt{5}}{2}$, x>0 and then x= $\frac{-1+\sqrt{5}}{2}$.

Therefore, in this particular case of parabolas we'll have a square inscribed in the curve.

Particular case 2



We consider $f: \mathbb{R} \to \mathbb{R}$, f(x) = x^2-a, $g: \mathbb{R} \to \mathbb{R}$ g(x) = $-x^2$+b, $a > 0, b > 0$.

To find the points where the graph of f and the graph of g intersects, we solve the equation f(x)=g(x) $\Leftrightarrow 2x^2=a+b \Leftrightarrow x=\pm\sqrt{\frac{a+b}{2}}$.

We consider m $\in (0, \sqrt{\frac{a+b}{2}})$ and A (m, g(m)), B(-m, g(-m)), C(-m, f(-m)) ,D(m, f(m)). ABCD is a rectangle.

AB=2m ; BC= g(-m)-f(-m)=$-x^2+b-x^2+a$

ABCD is a square $\Leftrightarrow 2m= a+b-2m^2 \Leftrightarrow 2m^2+2m-(a+b)=0$. $\Delta=4+8(a+b) > 0$ and $m_{1,2}=\frac{-1\pm\sqrt{1+2a+2b}}{2}$

$m> 0 \Leftrightarrow m=\frac{-1+\sqrt{1+2a+2b}}{2}$.

We check if $m <\sqrt{\frac{a+b}{2}} \Leftrightarrow -1+\sqrt{1+2a+2b}<\sqrt{2a+2b} \Leftrightarrow 1+2a+2b<2a+2b+2\sqrt{2a+2b}+1$, true.

Therefore, in this particular case of parabolas we'll have a square inscribed in the curve.

7. Conclusion

We found different situations of polygons and closed curves in which exist a square inscribed on them. We used geometric methods and the properties of quadratic functions to demonstrate the square exist. Almost all the closed curves used for our research had axis of symmetry, The only figure that does not have an axis of symmetry is the scalen triangle.