Flyovers

Year 2021– 2022

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1. PRESENTATION OF THE RESEARCH TOPIC

Connecting cities is essential and when this is done, intersections can be created. We want to present how we get the minimum number of overbridges for n towns so that intersections are avoided. In order to try to get the minimum number of overbridges, we tried to solve the problem for some particular cases.

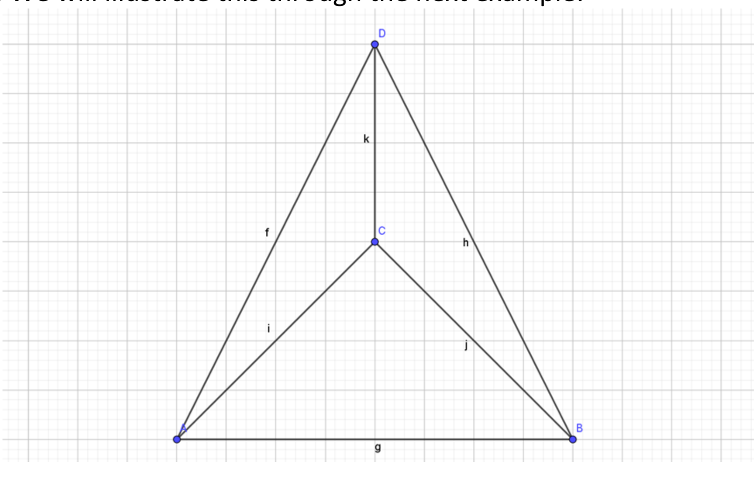
2. BRIEF PRESENTATION OF THE CONJECTURES AND RESULTS OBTAINED

A team of engineers has the task of building motorways connecting a number of towns (say n) in a certain region of the country. Any two of these towns must be connected by a single direct motorway. However, the motorways must not intersect each other. To achieve this, the engineers must construct a number of overbridges (an overbridge is a motorway bridge that crosses over another motorway). Can you help them find the minimum number of overbridges they must construct? Start with small values of n.

3. THE SOLUTION

Our first idea was to try to find a way to connect the points that can be easily generalized. One way to do this was to start with a triangle and to add points over the “highest edge” of the triangle. We considered that the number of flyovers is equal to the number of intersections by drawing the lines corresponding to the roads in the same plane.

Firstly, we found out that we don’t need any overbridge to connect four cities so that any two cities can be connected directly. We will illustrate this through the next example:



It is clear that we do not need any overbridge for 2 or 3 cities because we can chose 2 or 3 from the previous example and connect them in the same way.

When it comes to 5 points we proved the following result. We denote by f_n the number of overbridges needed to connect n towns.

For $\boldsymbol{n = 5}$**.**

Proposition 1. *The minimum number of flyovers connecting 5 towns is non-zero. In fact,* $f_5 = 1$.

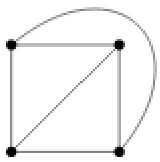
To prove the affirmation above, we will use Euler's formula:

For any planar graph having $\boldsymbol{V}$ *vertices,* $\boldsymbol{E}$ *edges and* $\boldsymbol{F}$ *faces, the following relation holds:*

$$\boldsymbol{V - E + F = 2}$$

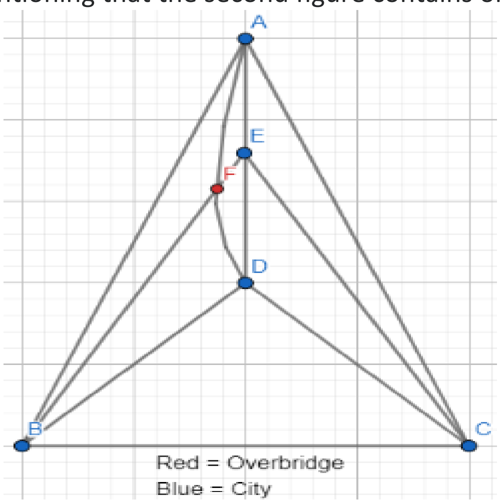
Definitions:

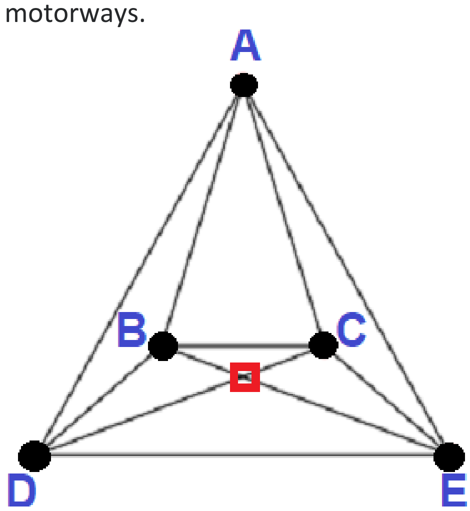
- A **graph** can be defined as a pictorial representation (or a diagram) of a structure made of vertices (or nodes = towns) and edges (connections between edges = motorways). Edges could be segments or curves.
- A **connected graph** is a graph for which there is a path from any point to any other point in the graph.
- A **planar graph** is a connected graph in the plane that can be drawn without edge crossings.
- When a planar graph is drawn without edges crossing, the edges and the vertices of the graph divide the plane into regions (some of them are infinite), which are called ***faces***. For example, the graph drawn below has 4 vertices, 6 edges and 4 faces ($4 - 6 + 4 = 2$).



The proof of the Proposition 1 is done by contradiction.
For f_5, we have $V = 5$ (one for each of the vertices), $E = 10$ (which is calculated by adding $4 + 3 + 2 + 1$). So assume that $f_5 = 0$ because we want to know if we can connect 5 points to one another with the minimal amount of overbridges 0**.** This means that we have a planar graph, and so, it must satisfy Euler's formula for planar graphs, $V - E + F = 5 - 10 + F = 2$, thus $F = 7$.
As all the faces must be triangular (otherwise, you could add a diagonal edge through the face (1)), we have $3F = 2E = 20$, and so $F = 20/3...$, contradiction.

We present 2 ways to arrange the points in order to obtain only one overbridge for 5 points. It is worth mentioning that the second figure contains only straight-line motorways.





$n > 5$

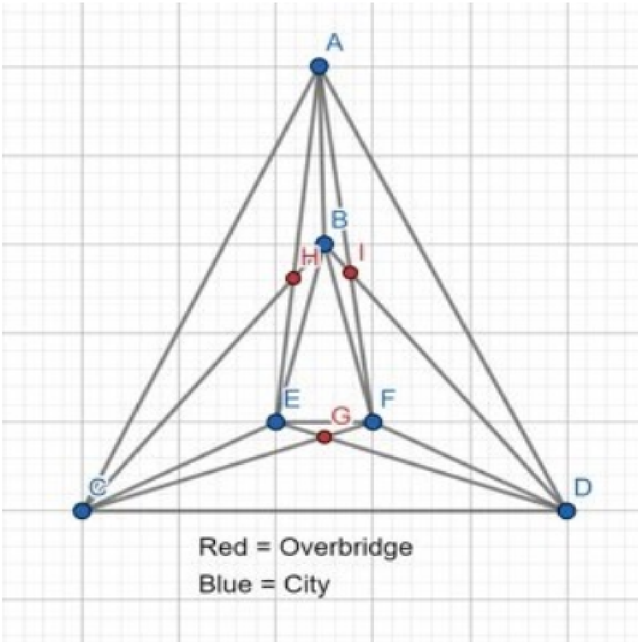
Proposition 2: *The minimum number of flyovers connecting $n > 5$ towns is non-zero.*

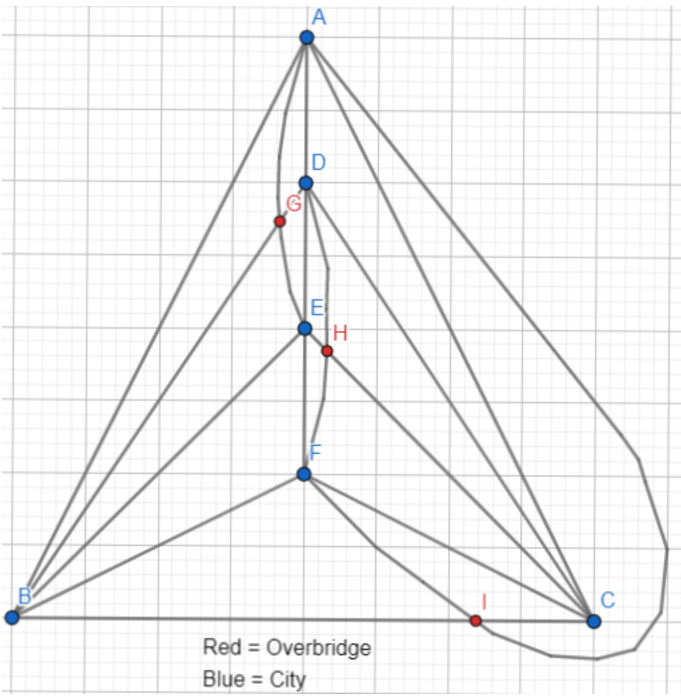
Proof: Indeed, if we suppose that for some $n \geq 6$ we have $f_n = 0$, then removing one or more vertices, the same property must hold. As $f_5 = 1 > 0$, we find this to be impossible.

$n = 6$

Proposition 3*: The minimum number of flyovers connecting $n = 6$ towns is* 3*.*

Proof: Firstly, from the figure drawn below, we conclude that $f_6 \leq 3$. Suppose there exists a drawing that has two flyovers. Both crossings involve four distinct vertices. Since the graph has six towns, there is at least one town (say X) that finds itself at the end of both flyovers. On the other side, if we remove town X, the resulting configuration would have no flyovers. This means that we could build a configuration of connected 5 towns with no flyovers, which is a contradiction to the demonstration from the previous point. Since we proved that we need at least 3 overbridges in order to connect 6 towns it is enough to find an example that shows that 3 overbridges are enough to connect 6 towns.

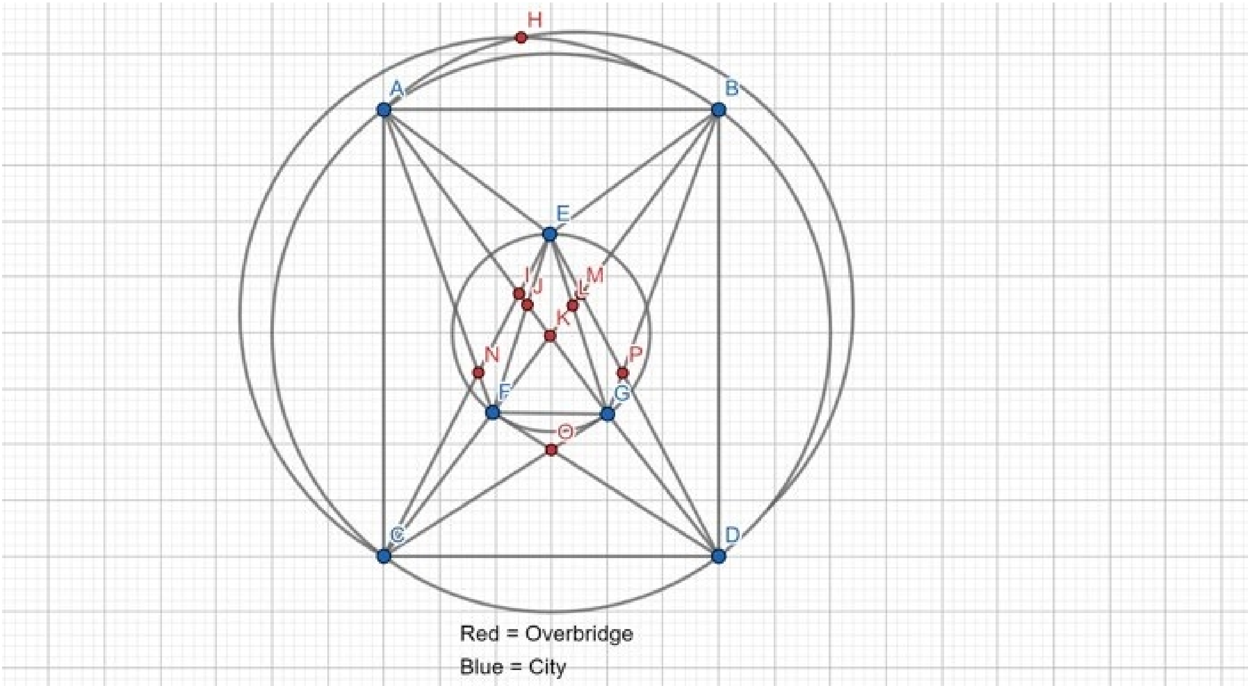


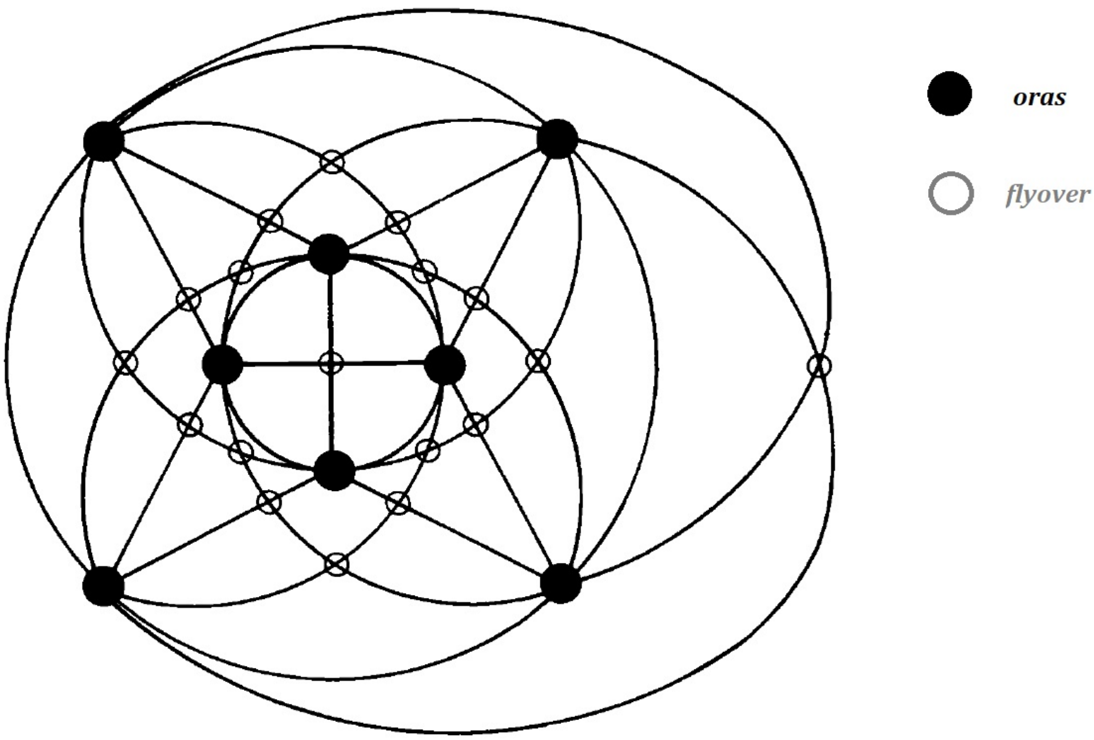


We tried to generalize some of the obtained configurations is order to find the number of flyovers for bigger numbers n.

The main idea was to arrange half of the points on a circle and half on the other (2). When n is even, we arrange the points on two concentric circles so that the number of points on a circle is equal to the number of points on the other circle. When n is odd, we arrange the points on two concentric circles so that a circle has a point more than the other. It does not matter if the bigger circle is the one that has the bigger number of points or not.

In this way we managed to find that the minimum number of flyovers for 7 towns is less than or equal to 9, and for 8 towns is not greater than 18. These results can be observed from the following pictures:





4. CONCLUSION

To solve this problem, we used a little bit of geometry and calculus. In order to find the minimum number of overbridges, we analyzed few particular cases to grasp an idea about how to approach the problem. We also managed to find out the minimum number of overbridges for 6 towns or less, and tried to generalize our ideas for more towns.

Notes d'édition

(1) This requires a brief explanation because, if we have a face with at least 4 vertices A, B, C, D in this order, it may happen that A and C are already connected by an edge e outside the face, so that we cannot add the diagonal AC. But in this case e and AC form a closed path that the diagonal BD intersects once. One of the points B and C is inside the path and the other outside, and a edge joining them would necessarily cut it. There can therefore not be an edge joining B and D outside the face and we can add the diagonal BD.

(2) It would have been nice to know what led to this idea and how it allows reducing the number of crossings.
However, a complete proof seems very difficult even for the cases $n = 7$ or $n = 8$, and out of reach for larger values of n. But, if it seems impossible to determine exactly the minimum number of crossings, one can ask the question of estimating its order of magnitude.