Two Problems on Touching Circles and Their Connection to the Stern-Brocot Tree

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Abstract

Sangaku are Japanese geometrical problems which were placed as offerings at Shinto shrines or Buddhist temples in Japan several centuries ago. The aim of our work is to study two of these Sangaku regarding different configurations of touching circles and to eventually show their connection. Essentially both problems ask to find a circle built using other circles, given a starting configuration.

Regarding the first problem we first show how to solve it and then, by infinitely iterating the initial geometrical configuration, we build a binary tree structure for the radii of all the generated circles. Showing that the tree structure for the radii is related to the Stern-Brocot tree (a binary tree expressing all positive rational numbers), we provide a method to find the radius of any circle given the radii of the initial two circles. We are also able to give a closed expression for the total area of a particular collection of circles.

Finally, using the properties of the inversion transformation we show how to map the first Sangaku to the second one highlighting the fact that they represent actually the same problem.

Contents

Introduction

The ancient Japanese culture included Wasan, the eastern mathematical method, over time replaced by yosan, or western mathematics. Wasan was practised by students but also by ordinary people who faced mathematical problems for fun or for needs of daily life.

The problems were drawn on wooden tables, called SANGAKU, which were hung in front of temples to challenge anyone who wanted to find and write the solution on the table. It was a very participatory and fun mathematics as perhaps it should also be today.

For the most part the sangaku dealt with Euclidean geometry topics, but the problems covered are different from those we are used to seeing in yosan geometry: circles and ellipses inscribed in each other are the masters, along with circular sectors. In most sangaku, only the problem and the result are provided, thus leaving us in complete ignorance of the method used to solve it.

This year we studied two Sangaku problems and in this article we are going to propose the way to reach the solution.

Problem 1. Given three circles tangent to each other and to a straight line, express the radius of the middle circle using the radii of the other two.
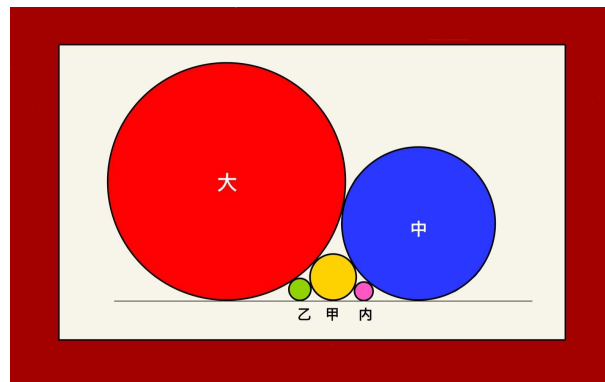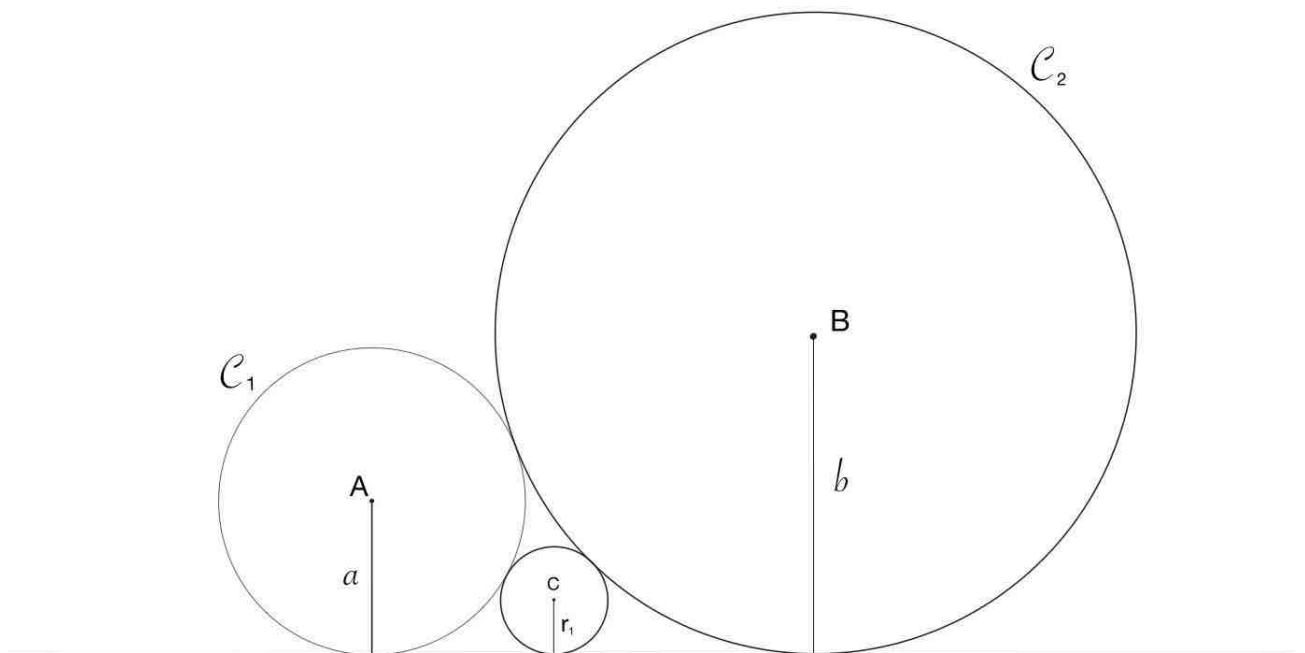

Problem 2. Given three circles inscribed inside a larger circle, express the radius of the middle circle using the radii of the other two.
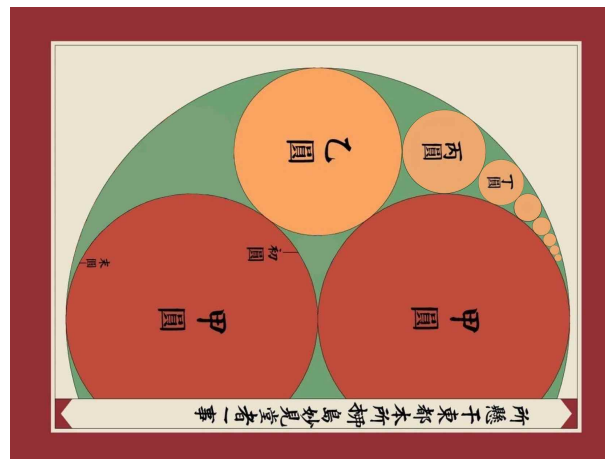
1 The first problem

Let's start by analysing the first problem, we call

- a, the radius of the circle C_1 on the left;
- b, the radius of the circle C_2 on the right;
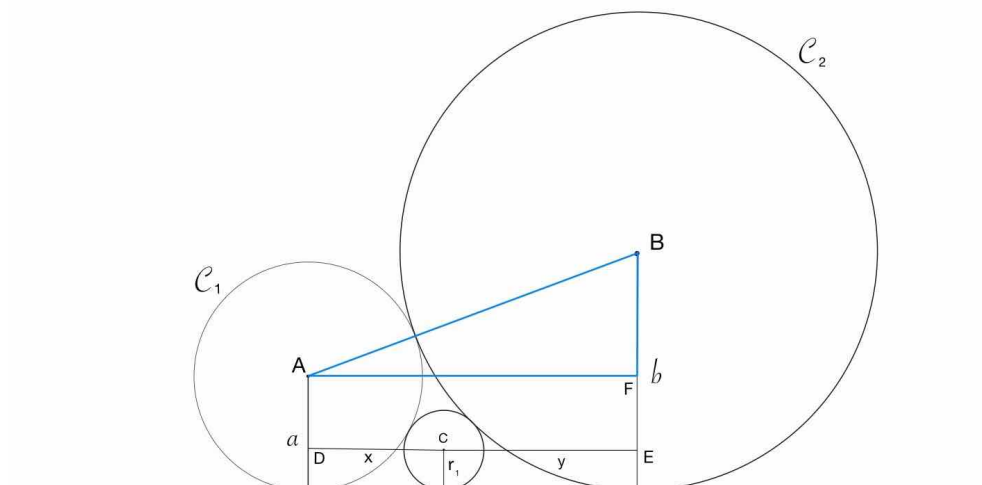- r_1 the radius of the middle circle.

Consider the right triangle ABF, whose hypotenuse AB is the sum of radii a and b. By the Pythagorean theorem, we know that:

$$AB^2 = AF^2 + BF^2$$

Since AF and BF are equal to $x + y$ and $b - a$ respectively, where $x = DC$ and $y = CE$, the equation eventually takes the form:

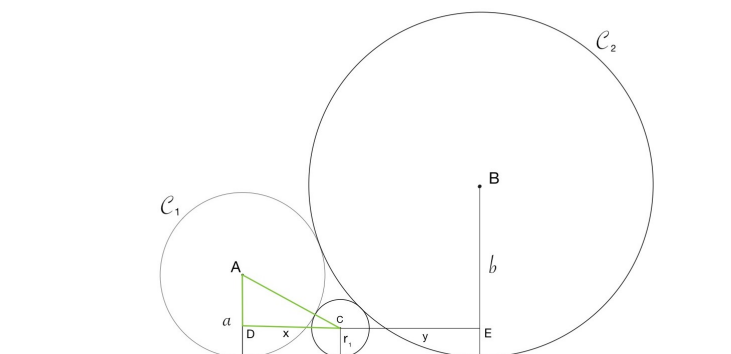$$(a + b)^2 = (x + y)^2 + (b - a)^2$$

Consider the right triangle ADC, whose hypotenuse AC is the sum of radii a and r_1. By the Pythagorean theorem, we know that:

$$AC^2 = DA^2 + DC^2$$

Since AD and DC are equal to $a - r_1$ and x respectively, the equation eventually takes the form:
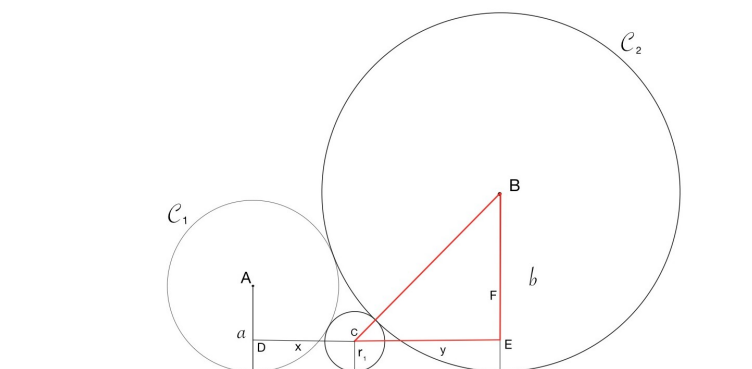
$$(a + r_1)^2 = (a - r_1)^2 + x^2$$

Consider the right triangle BEC, whose hypotenuse BC is the sum of b and r_1. By the Pythagorean theorem, we know that:

$$BC^2 = CE^2 + BE^2$$

Since BE and CE are equal to $b - r_1$ and y respectively, the equation eventually takes the form:

$$(b + r_1)^2 = y^2 + (b - r_1)^2$$

Therefore we get:

$$\begin{cases} (a + b)^2 = (x + y)^2 + (b - a)^2 \\ (a + r_1)^2 = (a - r_1)^2 + x^2 \\ (b + r_1)^2 = y^2 + (b - r_1)^2 \end{cases}$$

So, remembering that all quantities are positive,

$$\begin{cases} 4ab = x^2 + 2xy + y^2 \\ 4ar_1 = x^2 \\ 4br_1 = y^2 \end{cases}$$

From the second equation of the system we get $x = 2\sqrt{r_1 a}$ and from the third we get $y = 2\sqrt{r_1 b}$. Substituting in the first equation we obtain $ab = ar_1 + 2r_1\sqrt{ab} + br_1$.
Dividing by r_1 and factoring,

$$\frac{ab}{r_1} = (\sqrt{a} + \sqrt{b})^2 \qquad \text{and} \qquad \sqrt{\frac{ab}{r_1}} = \sqrt{a} + \sqrt{b},$$

and finally dividing by $\sqrt{ab}$ we get

$$\frac{1}{\sqrt{r_1}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

2   The iteration and the Stern-Brocot tree

After solving the first Sangaku we iterated the process by drawing other circles tangent to the 3 initials and looking for the relationship between the radii of the new circles and the radii a and b. Infinitely many circles are formed and we have ordered their radii as shown in the following figure.
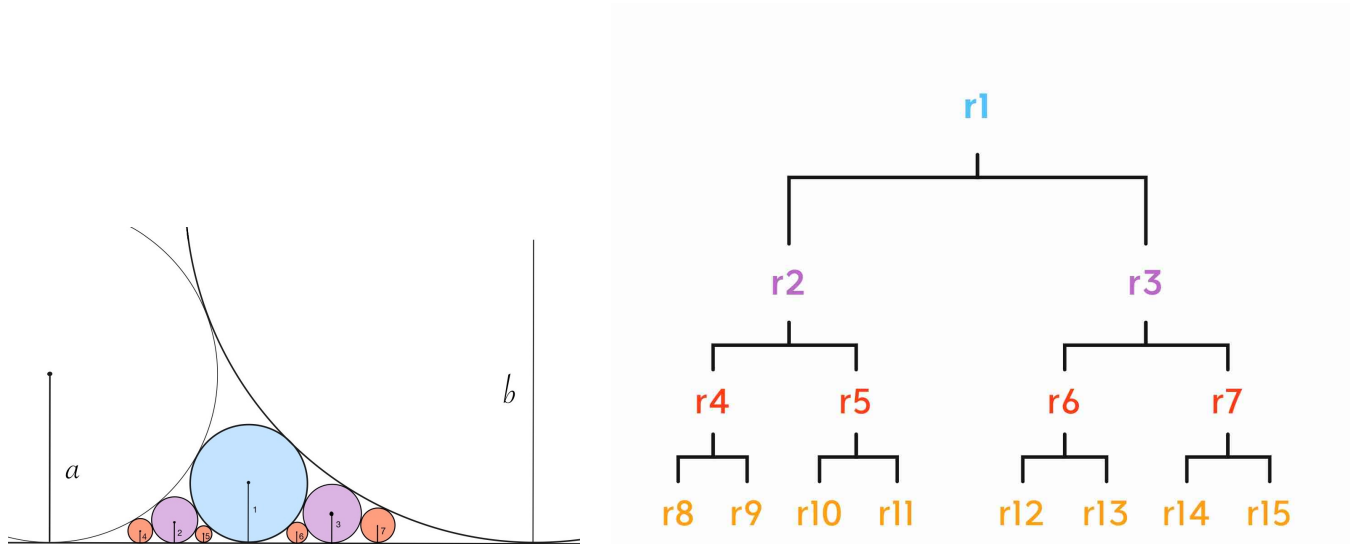
First of all we found the relationship

$$\frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{r_1}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{2}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

Then we rewrote the formula

$$\frac{1}{\sqrt{r_2}} = \frac{2}{\sqrt{a}} + \frac{1}{\sqrt{b}} = (2, 1)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix}$$

and in the same way we found:

$$\frac{1}{\sqrt{r_2}} = \frac{2}{\sqrt{a}} + \frac{1}{\sqrt{b}} = (2,1)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix} \qquad \frac{1}{\sqrt{r_9}} = \frac{5}{\sqrt{a}} + \frac{2}{\sqrt{b}} = (5,2)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix}$$

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{a}} + \frac{2}{\sqrt{b}} = (1,2)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix} \qquad \frac{1}{\sqrt{r_{10}}} = \frac{5}{\sqrt{a}} + \frac{3}{\sqrt{b}} = (5,3)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix}$$

$$\frac{1}{\sqrt{r_4}} = \frac{3}{\sqrt{a}} + \frac{1}{\sqrt{b}} = (3,1)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix} \qquad \frac{1}{\sqrt{r_{11}}} = \frac{4}{\sqrt{a}} + \frac{3}{\sqrt{b}} = (4,3)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix}$$

$$\frac{1}{\sqrt{r_5}} = \frac{3}{\sqrt{a}} + \frac{2}{\sqrt{b}} = (3,2)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix} \qquad \frac{1}{\sqrt{r_{12}}} = \frac{3}{\sqrt{a}} + \frac{4}{\sqrt{b}} = (3,4)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix}$$

$$\frac{1}{\sqrt{r_6}} = \frac{2}{\sqrt{a}} + \frac{3}{\sqrt{b}} = (2,3)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix} \qquad \frac{1}{\sqrt{r_{13}}} = \frac{3}{\sqrt{a}} + \frac{5}{\sqrt{b}} = (3,5)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix}$$

$$\frac{1}{\sqrt{r_7}} = \frac{1}{\sqrt{a}} + \frac{3}{\sqrt{b}} = (1,3)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix} \qquad \frac{1}{\sqrt{r_{14}}} = \frac{2}{\sqrt{a}} + \frac{5}{\sqrt{b}} = (2,5)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix}$$
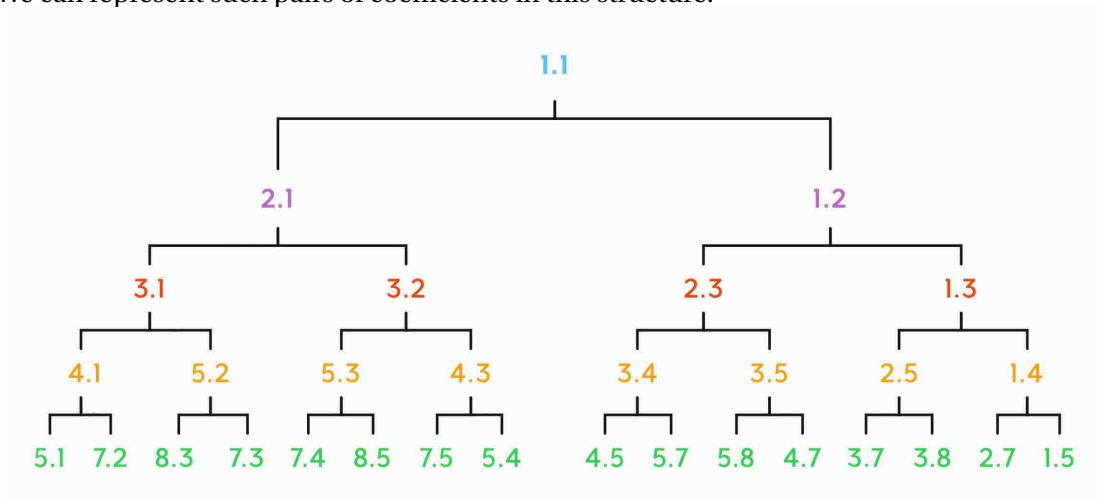
$$\frac{1}{\sqrt{r_8}} = \frac{4}{\sqrt{a}} + \frac{1}{\sqrt{b}} = (4,1)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix} \qquad \frac{1}{\sqrt{r_{15}}} = \frac{1}{\sqrt{a}} + \frac{4}{\sqrt{b}} = (1,4)\begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix}$$

Therefore for each new circle C_n there is a pair of coefficients (p,q), which allow to build a relationship between the radius r_n of the circle of index n and the radii a and b of the two initial circles:

$$\frac{1}{\sqrt{r_n}} = \frac{p}{\sqrt{a}} + \frac{q}{\sqrt{b}}$$

We can represent such pairs of coefficients in this structure:

We noticed that the structure we got looks like the tree of Stern-Brocot with only one difference, which is the inverted order of the elements.
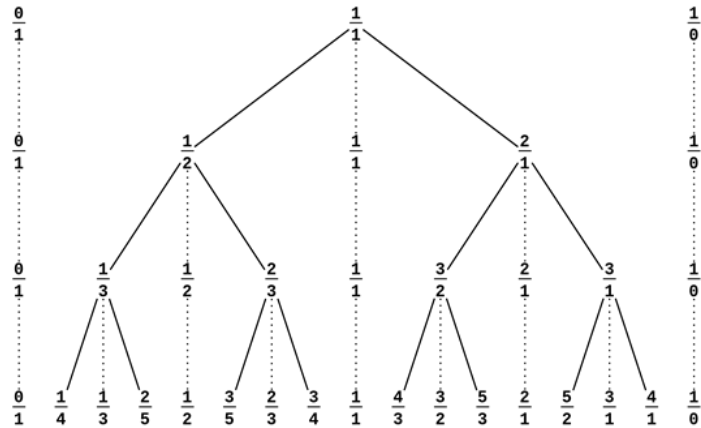

The SternBrocot tree is an infinite binary tree in which the vertices correspond one-to-one to the positive rational numbers. It was introduced independently by Moritz Stern, a German number theorist, and Achille Brocot, a French clock-maker.

The tree is an enumeration of the positive rational numbers, it contains each positive rational number exactly once. Each new element in a new row is the mediant of two adjacent elements in the previous row. If a/b and c/d are two elements in the row i then the mediant $(a+c)/(b+d)$ is an element of the row $i+1$.

Moreover, two elements $\dfrac{a}{b} < \dfrac{c}{d}$ in the tree are adjacent iff $ad - bc = -1$. In order to study our tree we used the properties of the Stern-Brocot tree (1).

3 Relationship between the index of a circle and the pair of coefficients

In this section we are going to find a relationship between the index k of a circle and the pair of coefficients (p,q).
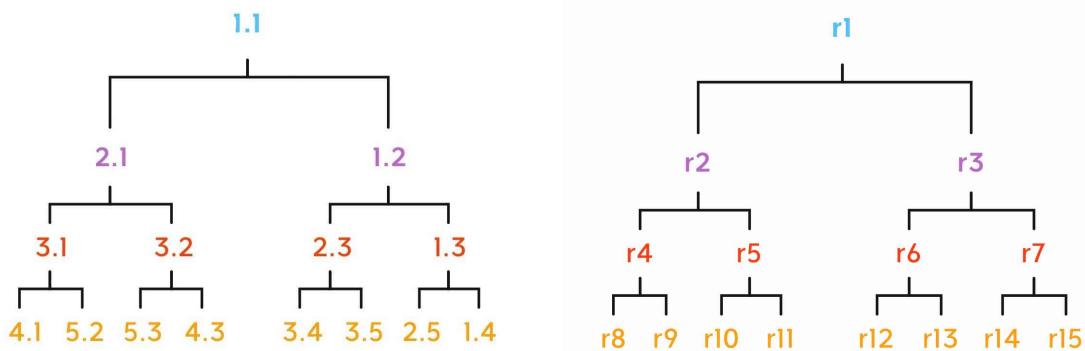
First of all we analysed the circles on the extreme left and on the extreme right and we got for the row n the following formulas:
For the circle on the extreme right,

$$\frac{1}{\sqrt{r_{2^{n+1}-1}}} = \frac{1}{\sqrt{a}} + \frac{n+1}{\sqrt{b}} = (1, n+1) \begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix}$$

Example: For the forth row ($n = 3$):

$$\frac{1}{\sqrt{r_{15}}} = \frac{1}{\sqrt{a}} + \frac{4}{\sqrt{b}} = (1,4) \begin{pmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \end{pmatrix}$$

For the circle on the extreme left,

$$\frac{1}{\sqrt{r_{2^n}}} = \frac{n+1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = (n+1,1)\begin{pmatrix}\frac{1}{\sqrt{a}}\\\frac{1}{\sqrt{b}}\end{pmatrix}$$

Example: For the forth row ($n = 3$):

$$\frac{1}{\sqrt{r_8}} = \frac{4}{\sqrt{a}} + \frac{1}{\sqrt{b}} = (4,1)\begin{pmatrix}\frac{1}{\sqrt{a}}\\\frac{1}{\sqrt{b}}\end{pmatrix}$$

After we chose a random circle in the middle of the tree and we looked for the formula that gives the index of the circle get the coefficient $r_k \rightarrow (p,q)$ and vice-versa $(p,q) \rightarrow r_k$.

Let's see it with an example.

Example: Given the circle number 39, we transform the index into a binary number:

$$39_2 = 100111$$

By removing the first value (the top of the tree) we get the number that represent the path to reach the circle, 0 corresponds to a movement to the left and 1 corresponds to a movement to the right (2):

$$00111 = LLRRR$$

Then we assign to L and R these matrices:

$$L = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$$

$$R = \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}$$

We substitute:

$$LLRRR = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}$$

Calculating we get:

$$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}^2\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}^3 = \begin{pmatrix}7 & 2\\3 & 1\end{pmatrix} \quad \text{which yields } (7,3) \text{ and}(2,1), \text{ that is } (9,4).$$

And those are the coefficients (p,q) of the circle C_{39}:

$$\frac{1}{\sqrt{r_{39}}} = (9,4)\begin{pmatrix}\frac{1}{\sqrt{a}}\\\frac{1}{\sqrt{b}}\end{pmatrix}$$

We assign to 0 and 1 two matrices because we noticed that, given two adjacent elements (p,q) and (m,n) on a row i of the tree, in row $i+1$ we add in the middle the mediant $(p+m,q+n)$ and so the next couple of elements is either (p,q) and $(p+m,q+n)$ or $(p+m,q+n)$ and (m,n).

Given (p,q) and (m,n) to get (p,q) and $(p+m,q+n)$ we multiply by the matrix L and to get $(p+m,q+n)$ and (m,n) we multiply by the matrix R.

$$(p,q) \quad (m,n)$$
$$(p,q) \quad (p+m,q+n) \quad (m,n)$$

$$\begin{pmatrix} p & m \\ q & n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & p+m \\ q & q+n \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = L$$

$$\begin{pmatrix} p & m \\ q & n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} p+m & m \\ q+n & n \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = R$$
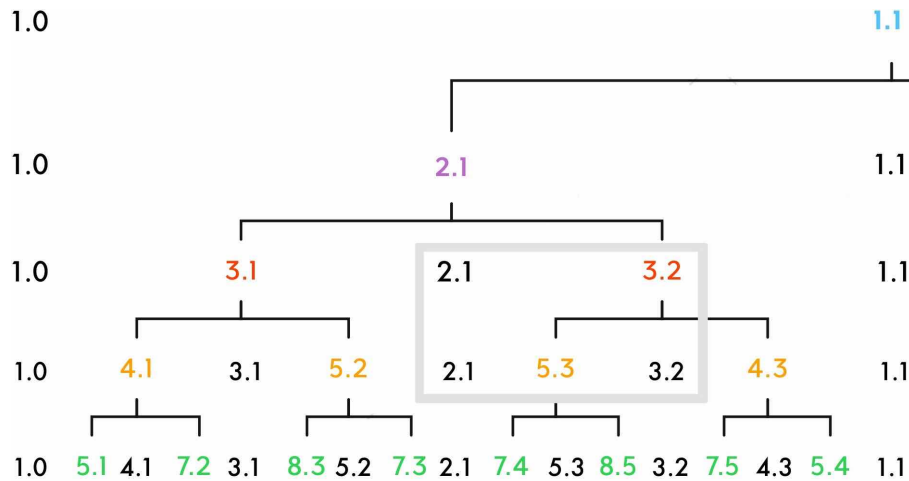
After founding the formula that gives the index get the coefficients, we analysed the opposite case $(p,q) \to r_k$.

Example: Given the coefficients $(9,4)$, we started looking for the parents (p,q) and (m,n) of the pair.

We use the Stern-Brocot tree properties that state that $(9,4)$ is a mediant, that is $p+m=9$ and $q+n=4$, and that $pn-qm=1$.

Thanks to the the first propriety we find the possible pairs:

$$9 = 8+1 = 7+2 = 6+3 = 5+4, \qquad 4 = 3+1 = 2+2$$

in the Stern-Brocot tree there is only one possibility and so it is $7+2$ and $3+1$ (3). Then

$$(7,3) \quad (9,4) \quad (2,1)$$

Now we consider the coefficients $(7,3)(2,1)$ of the previous row and we looking for the missing parent. We notice that $(7,3)$ is the mediant so we find the missing LEFT parent $(5,2)$.
So we get

$$(5,2) \quad (7,3) \quad (2,1)$$

We repeat the procedure find the missing LEFT parent $(3,1)$ and we get

$$(3,1) \quad (5,2) \quad (2,1)$$

We repeat the procedure find the missing LEFT parent $(1,0)$ and we get

$$(1,0) \quad (3,1) \quad (2,1)$$

We repeat the procedure find the missing RIGHT parent $(1,1)$ and we get

$$(1,0) \quad (2,1) \quad (1,1)$$

We repeat the procedure find the missing RIGHT parent $(0,1)$ and we get the top of the tree

$$(1,0) \quad (1,1) \quad (0,1)$$

We started from circle 39 and went up, then going down we have to reverse the right and left movements that is $LLRRR$ therefore we get the number

$$1LLRRR = 100111 = 39_2$$

4   Fibonacci and Lucas numbers

Building and studying our tree we noticed the presence of the Fibonacci numbers and the numbers of a less known series that is Lucas numbers.

These numbers can be found both in the right part of the tree and in the left part since the two parts are symmetrical.
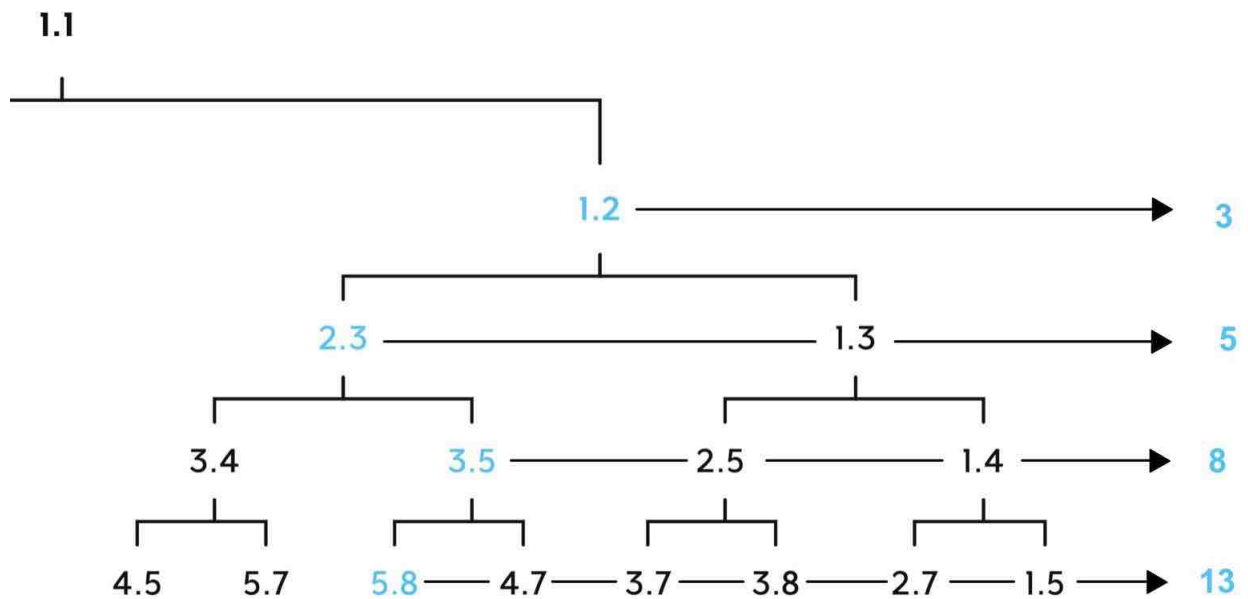
Fibonacci numbers The Fibonacci numbers may be defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1$$

and

$$F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n > 1$$

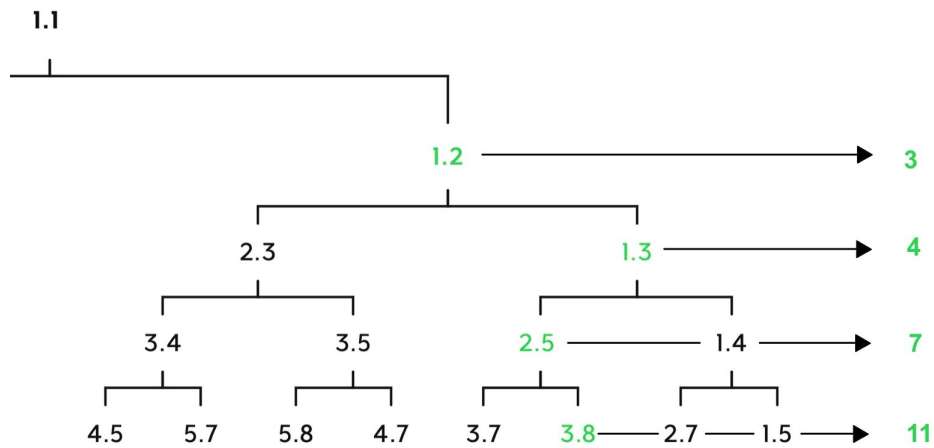the first are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

Lucas numbers The Lucas numbers may be defined by the recurrence relation

$$L_0 = 2, \quad L_1 = 1$$

and

$$L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n > 1$$

the first are 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, …

1.1

1.2 ——————————→ 3

2.3   1.3 ——————→ 4

3.4   3.5   2.5 ————— 1.4 ——→ 7

4.5   5.7   5.8   4.7   3.7   3.8——2.7——1.5 —→ 11

5 Analysis of some properties of a subset of circles

In this section we study some aspects of a particular subset of circles generated starting from the two main ones. Fig. 1 and its description recalls the characteristics of one of the two Sangaku problem we studied.
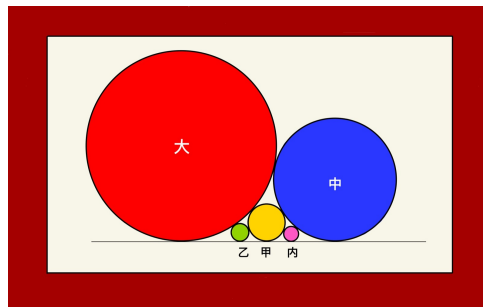

Figure 1: In the original Sangaku problem the red and blue circles are placed tangent to each other. The goal is to find the radius of the yellow circle. Following this reasoning the process can be iterated to find the radii of the green and pink circles and so on.

We recall the geometrical problem in the following picture. Given a circle $\mathscr{A}$ with radius a and center $A = (0; a)$ let's call $\mathscr{B}$ the circle tangent to $\mathscr{A}$ and the x-axis, whose center B will have coordinates $(x_B; b)$. As shown in Fig. 2, applying the Pythagora's theorem to the right triangle whose hypotenuse is the segment between the centers A and B and whose catheti are parallel to the x and y axes, we get:
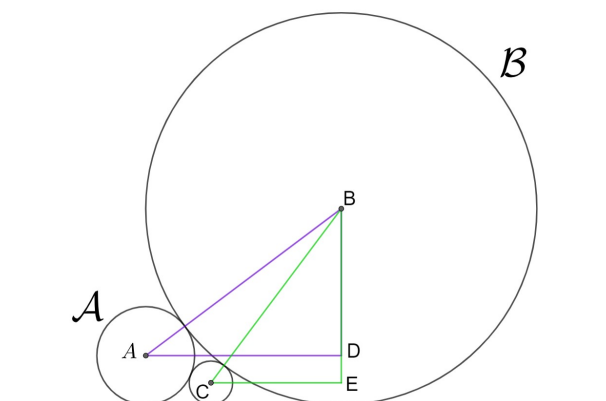
$$x_B = 2\sqrt{ab} \tag{1}$$

Figure 2: Pythagora's theorem applied to the right triangles constructed with the vertices A, B, D and with the vertices B, C, E

Now we first draw the first middle circle between $\mathscr{A}$ and $\mathscr{B}$ and then we iterate the construction to obtain two chains of circles as shown in Fig. 3, both on the right and on the left of the middle circle.
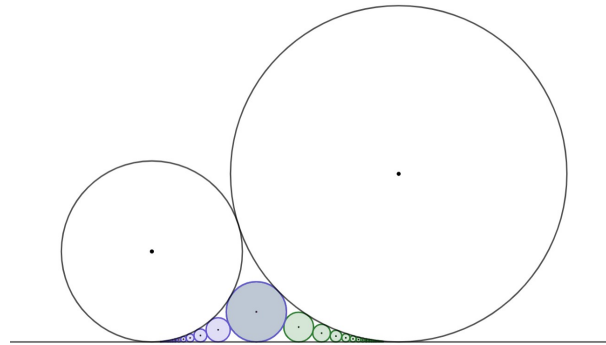

Figure 3: Left and right chains of circles.

According to the tree structure reported in Fig. 4 the rightmost chain correspond to the rightmost branch of the tree (see Sec. 2 for a detailed explanation of the origin of the tree of indices).
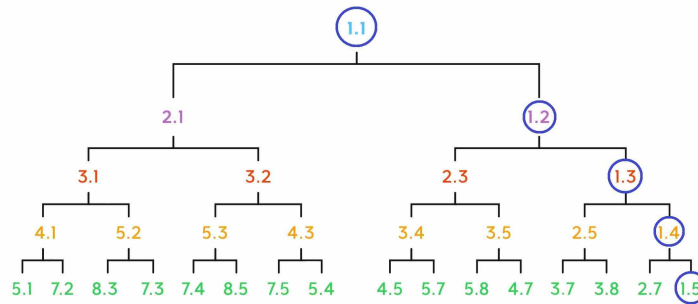

Figure 4: Tree structure to show the radii of interest. In the following we analyse only the rightmost chain of circles indexed by the highlighted pair of integers. For all the details about the tree structure and the meaning of its nodes see Sec. 2

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In the following we're going to analyse some properties of these collection of circles. To avoid repeating similar equations to describe the properties for the left and the right chain of circles we will focus only on the right one. The centers of these circles will have coordinates $C_n = (x_n; r_n)$, where x_n and r_n are given by the following expressions

$$r_n = \frac{b}{\left(n + \sqrt{\dfrac{b}{a}}\right)^2}$$

$$x_n = x_B - 2\sqrt{br_n} \tag{2}$$

The expression for x_n has been obtained applying a method similar to the one used to derive Equation (1). The expression for r_n instead follows from the general expression for the radii (see Sec. 2 for the details). We here recall that

$$\frac{1}{\sqrt{r_n}} = \frac{p}{\sqrt{a}} + \frac{q}{\sqrt{b}}$$

where (p, q) identifies the circle under examination inside the tree structure (see Fig. 4). Thus the expression for the radius of the n-th circle of the rightmost collection corresponds to $(p, q) = (1, n)$ thus giving the first equation in Equation (2).

5.1 Parabola through the centers

As shown by the following geometrical proof, all the centers of the circles lie on a parabola. As shown in Fig. 5 each point has the same distance from the point B and the line of equation $y = -b$, which are, respectively, the focus and the directrix of the parabola (4).
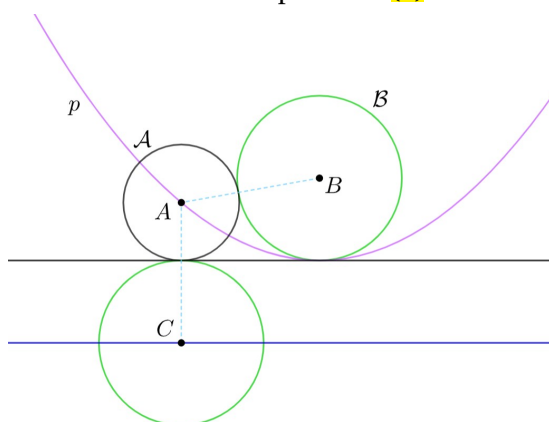

Figure 5: Picture that shows how the theorem works

With some calculations we can obtain the equations of these two parabolas. In particular the parabola passing through the centers of the right chain of circles can be obtained manipulating the expression for the coordinates of the centers given in Equation (2).

$$\begin{cases} x = x_b - \dfrac{2b}{n + \sqrt{\frac{b}{a}}} \\[3ex] y = r_n = \dfrac{b}{\left(n + \sqrt{\frac{b}{a}}\right)^2} \end{cases} \tag{3}$$

showing that

$$y = \frac{(x_b - x)^2}{4b} \tag{4}$$

The same reasoning applies to the left chain of circles. In fact the coordinates of the centers are given by

$$\begin{cases} x = \dfrac{2a}{n + \sqrt{\frac{a}{b}}} \\[3ex] y = r_n = \dfrac{a}{\left(n + \sqrt{\frac{a}{b}}\right)^2} \end{cases} \tag{5}$$

Which simplifies to:
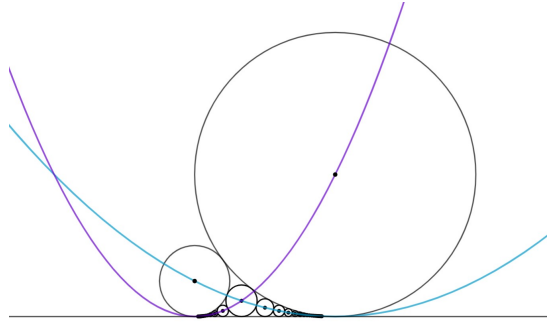
$$y = \frac{x^2}{4a} \tag{6}$$

Figure 6: The centers of the circles lay on two different parabolas

5.2 Total area of the circles

We now show how to give a closed expression for the total area of all the circles in the right chain, which we will call $Z(a, b)$. The expression is obtained by substituting the expression of the radius (Equation (2)) into the expression for the area of a circle and adding up all the values obtained for n going from 0 (which refers to the circle $\mathscr{A}$) to ∞:

$$Z(a, b) = \pi b^2 \sum_{n=0}^{\infty} \frac{1}{(n + \sqrt{\frac{b}{a}})^4} \tag{7}$$

To our surprise he summation part defines what is called the Hurwitz's zeta function $\zeta\left(\sqrt{\frac{b}{a}}, 4\right)$. In this way the expression for the total area can be rewritten in compact form as

$$Z(a, b) = \pi b^2 \zeta\left(\sqrt{\frac{b}{a}}, 4\right) \tag{8}$$

Being curious about the function's trend, we analysed some special cases: The first one is the case in which $a = b$. From Eq. 8 we get $\zeta(1, 4)$, which is precisely equal to the Riemann Zeta function of 4. For this reason the total area is equal to:

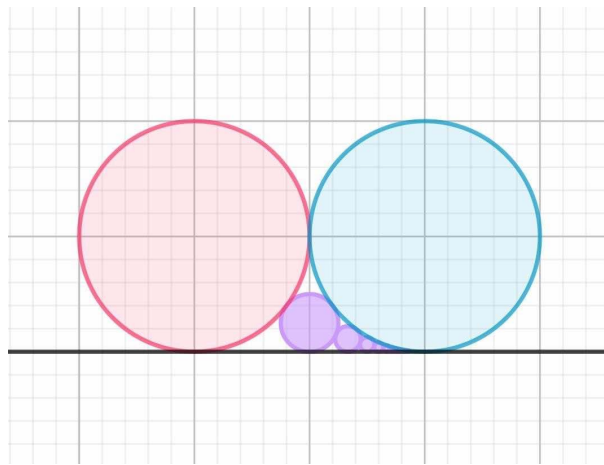$$A_{tot} = \pi b^2 \zeta(4) = \frac{\pi^5}{90} b^2 \quad \text{(5)}$$

Figure 7: The total area of the chain of circles between the two bigger ones can be computed exactly and is equal to $A_{tot} = \pi b^2 \zeta(4) = \frac{\pi^5}{90} b^2$

.

Another interesting case occurs when $b = n^2 a$, then $\sqrt{\frac{b}{a}} = n$ and the function takes the form:

$$\zeta(n,4) = \sum_{i=0}^{\infty} \frac{1}{(i+n)^4} \xrightarrow{i+n \longrightarrow i} \zeta(n,4) = \sum_{i=n}^{\infty} \frac{1}{i^4} \tag{9}$$

The last term can be rewritten using the Riemann zeta function of 4 minus a certain term.

For example, if we choose $n = 2$ we obtain:

$$\sum_{i=0}^{\infty} \frac{1}{(i+2)^4} = \sum_{i=2}^{\infty} \frac{1}{i^4} \tag{10}$$

The terms of the sum are $\frac{1}{2^4}, \frac{1}{3^4}, \frac{1}{4^4}, \ldots$

If we add 1 we get the Riemann zeta function of 4, so we can rewrite:

$$\sum_{i=2}^{\infty} \frac{1}{i^4} = \zeta(4) - 1$$

For a given n in the relation $b = n^2 a$, we can write:

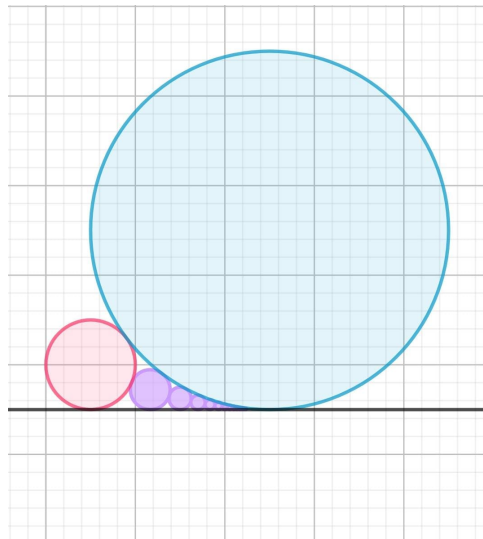$$\zeta(n,4) = \zeta(4) - \sum_{i=1}^{n-1} \frac{1}{i^4} \tag{11}$$

Figure 8: The case $b = 4a$ for which the total area of the chain of smaller circles can be exactly calculated as $\pi b^2 \zeta(2,4)$. Refer to the main text for the details.

6 Connecting the two Sangaku problems through inversion

After solving the first problem, we went on to analyse the second one and we were very surprised to find out that it is strictly related to the first one thanks to a particular geometric transformation, known as inversion about a circle. Let's find out how it works. It is a geometric transformation that maps each point of the plane to a new one such that the product of their distances from the centre of the inversion circle is equal to its radius squared.
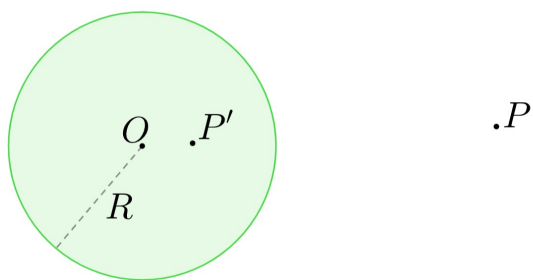
Figure 9: **Inversion of a point**: the point P is sent to P' according to equation $OP \cdot OP' = R^2$

The transformation of P to P' is defined by the following relation:

$$OP \cdot OP' = R^2$$

In this picture, you can see how to invert a line, which gets mapped into a circle, unless it passes through the centre of the inversion circle.
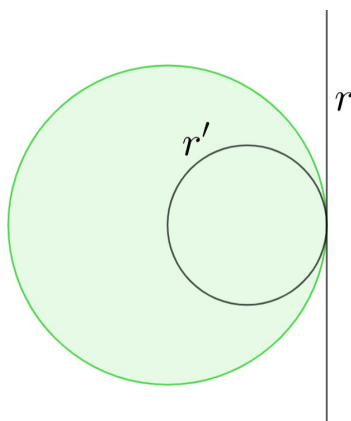

Figure 10: **Inversion of a line**: the inversion of the line r with respect to the green circle gives the circle r' and vice versa.

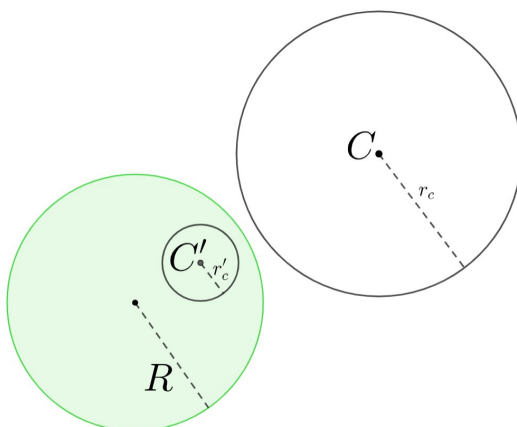In this other image, you can see how to invert a circle, which gets mapped into a new circle (6).

Figure 11: **Inversion of a circle:** the inversion with respect to the green circle maps the circle centered in C to the circle centered in C'

The new circle has radius $r_{C'}$ and center $C' = (x_{C'}, y_{C'})$ given by the following relations:

$$x_{C'} = \frac{R^2}{x_C^2 + y_C^2 - r_C^2} \, x_C$$

$$y_{C'} = \frac{R^2}{x_C^2 + y_C^2 - r_C^2} \, y_C$$

The new radius will be given by:

$$r_{C'} = \frac{R^2}{|x_C^2 + y_C^2 - r_C^2|} \, r_C$$

By then applying this transformation to the first Sangaku problem, we were able to map it into the second one.
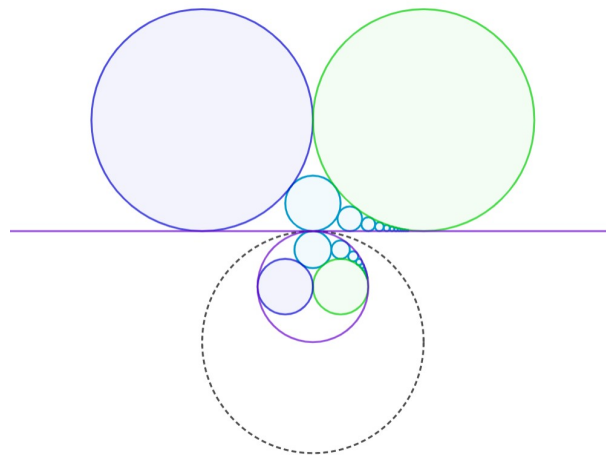

Figure 12: Inversion applied to our problem: we applied the inversion with respect to the dashed circle. The horizontal line is inverted to the purple circle.
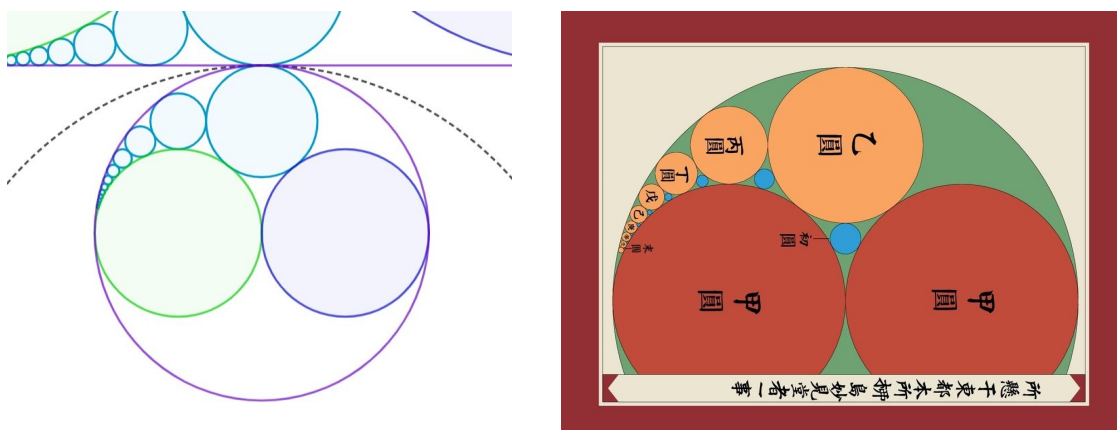

Figure 13: The two problems compared. Left picture: magnification of Fig.12. Right picture: the second Sangaku problem (see pag. 2 for a description of both the problems)

As we can see from the zoomed in picture, if we invert the whole set of circles with respect to the dotted circumference, the x-axis, represented by the purple line, gets mapped into the a circle containing the new set of tangent circles.

Editing Notes

(1) More information on the Stern-Brocot tree and proofs of these properties can be found e.g. on the Wikipedia page *Stern-Brocot tree* or in the MATh.en.JEANS article *Calendrier, fractions, Méthode de Brocot*.

(2) Here, pairs $(p, q), (m, n)$ of adjacent elements of the tree are considered, starting with $(1, 0)$ and $(0, 1)$. A movement to the left consists in replacing the right element by the mediant $(p + m, q + n)$, and a movement to the right consists in replacing the left element by the mediant. This corresponds to matrix multiplications as explained a few lines below. The final result is the mediant of the last pair.

(3) In general, finding the parent pair is not so easy (and other algorithms can be found in the above references). Note also that the order of the parents is given by $pn - qm = 1$.

(4) In figure 5, the circle $\mathscr{A}$ with radius a can be any circle tangent both to the circle $\mathscr{B}$ and to the x-axis; the circle centered at C is tangent from below to the x-axis at the same point as $\mathscr{A}$ and has same radius b as $\mathscr{B}$. So, we see that $AC = a + b = AB$.

(5) The value of $\zeta(4)$ (as well as that of $\zeta(2n)$ for every positive integer n) is a classical result due to Leonhard Euler.

Note that, as n is summed from 0 to $+\infty$, this total area includes that of the first circle $\mathscr{A}$. So, in figures 7 and 8, the total calculated area includes the areas of the red circles.

(6) A circle is mapped to a circle unless it passes through the centre of the inversion circle, in which case it is mapped to a line. A proof of these facts can be found e.g. on *Xah Lee's website*.