The Modelling of Fish Stocks

2021 - 2022

Diana Harambaș, Bianca Crișan (12th grade), Petru Săveanu (10th grade), Hugo Giret (11th grade), Loïc Brismoutier (9th grade)

Schools: Colegiul Național "Emil Racoviță" (Cluj), Lycée Val-de-Durance (Pertuis), Ensemble Scolaire Bellevue (Alès)

Teachers: Ariana Văcărețu, Hubert Proal, Sébastien Castagnedoli

Researchers: Serge Dumont (Université de Nîmes), George Țurcaș, ("Babeş-Bolyai" University)

Contents

1 Research Topic

In order to model the management of fish stocks, we consider the deterministic Schaefer model (1954). For that, we take into account a variable M (in tons) which represents the maximum biomass that can live in a certain site and a variable r which represents an intrinsic rate of growth (specific to each species of fish).

If we consider, for $n \geq 1$ the variable X_n, which is the biomass for the year n, the model gives the biomass for the year $n + 1$:

$$X_{n+1} = X_n + r\,X_n\left(1 - \frac{X_n}{M}\right) - C,$$

where C is the amount fished by humans (1).

Let X_0 be the initial biomass (given by a measurement).

Can we model, by modifying the parameters (initial biomass, the rate of reproduction, amount of fishing in a year), all the possible situations (extinction, uncontrolled growth of the species)?

2 Useful Tools

For the first approach, we used a C++ algorithm to help get a general idea of the problem. The program below shows the biomass value for all the years until extinction (2), after reading as input: the maximum biomass (M), the growth rate (r), the fished quantity (C), and the initial biomass (X_0).

```cpp
int M,C;
double r;
double x[1000];

void nextgen( int i )
{
    x[i] = x[i-1] + (r * x[i-1] * (1 - x[i-1]/M)) - C;
}

int main()
{
    cin >> M >> r >> C >> x[0];
    int i=0;
    while( x[i] > 0 )
    {
        i++;
        nextgen(i);
        cout << " " << x[i];
    }
    cout << i;
    return 0;
}
```

3 Solution

3.1 First Approach

This approach belongs to Diana Harambaș. It was done with the help of Bianca Crișan.

3.1.1 Extinction

The first question we asked ourselves was regarding when will the species go extinct. Hence, we calculated the value of X_n for which X_{n+1} is 0, i.e. we have extinction at the $(n+1)^{th}$ generation.

Solving the following second degree equation $(1+r)x - \frac{r}{M}x^2 - C = 0$, we obtain the roots:

$$\frac{(1+r) + \sqrt{(1+r)^2 - \frac{4rC}{M}}}{\frac{2r}{M}} \quad \text{and} \quad \frac{(1+r) - \sqrt{(1+r)^2 - \frac{4rC}{M}}}{\frac{2r}{M}}.$$

Obviously, the biomass is positive or null, but never negative. Hence, we need the positive root of the equation. Luckily, both solutions turned out to be positive, as $1 + r > \sqrt{(1+r)^2 - \frac{4rC}{M}}$. If there is an n for which X_n has one of the values above or is is outside the interval between these two values, then the next year the species will go extinct (when the next value is negative, there is still extinction).

3.1.2 Particular case: $C = 0$

This is the case where no fishing is done and the natural growth of the species is followed. When representing the growth on Geogebra, the majority of graphs (but not all) converge to M, the maximum biomass. See Figure 1 below:

FIGURE 1
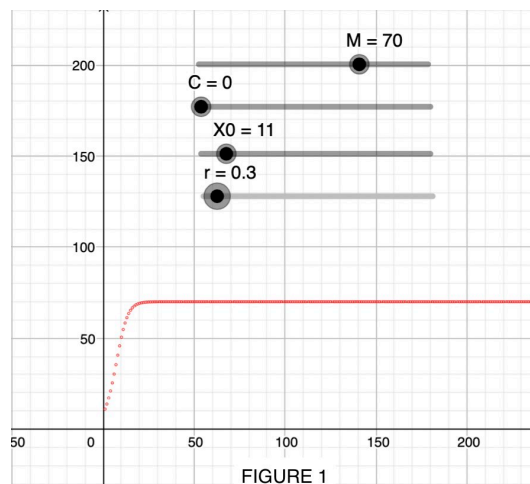
Firstly, let us study the variation and possible equilibrium points of this case. Between generation t and $t+1$ we have a biomass variation of $X_{t+1} - X_t = rX_t\left(1 - \frac{X_t}{M}\right)$. Let such a variation be represented as:

$$\Delta x = \frac{dx}{dt} = rx\left(1 - \frac{x}{M}\right)$$

Remark: Considering the differential equation, we move to a continuous time model. But the results may be very different from those in the discrete time model.

We can find the equilibrium points by solving $\Delta x = 0$. Thus, 0 and M are the equilibrium points. This means that for any t, if X_t is 0 or M, then $X_n = X_t$ for all $n \geq t$.

Knowing that M is the maximum biomass that can be in the lake, we can state that $x \leq M$, so $\frac{x}{M} \leq 1$ and $\Delta x = rx(1-\frac{x}{M}) \geq 0$ (in fact this is guaranteed by the model only for certain values of r - not for large values). As the variation is always positive, we can affirm that the sequence $(X_n)_{n \geq 1}$ is ascending (or constant). For any n, it is known that $0 \leq X_n \leq M$, which gives the last necessary condition for stating that $(X_n)_{n \geq 1}$ is convergent. Obviously, the sequence must converge to an equilibrium point, so to either M or 0. Given that $(X_n)_{n \geq 1}$ is ascending, we conclude that

$$\lim_{n \to \infty} X_n = M.$$

So, in a case where no fishing is done, if a fish species starts out with any strictly positive initial biomass, after many generations, the yearly biomass will end up as close to M as possible.

The question one has to ask oneself is: Is this biologically accurate? We have to consider environmental parameters as well, i.e. the lake dimension, the water volume, but most importantly the growth rate of the edible resources. In an ideal case, where the food resources follows a recurrence relation similar to the fish species one and has a greater growth rate, then there will forever be enough food for the fish stock. But, in reality, this might not be the case.

Formula: The differential equation $dx/dt = rx\,(1 - x/M)$ allows us to calculate the value of X_t for all $t \geq 1$ by using basic integration rules. Thus,

$$X_t = \frac{MX_0\,e^{rt}}{M + X_0\,(e^{rt} - 1)} \quad \text{for all } t \geq 1.$$

This is the solution for the continuous time model. Then we have always convergence towards M (except when $X_0 = 0$), whereas a periodic or cahotic variation may appear in the discrete model, as shown below.

Remark: We have considered a small rate r that does not let our fish biomass exceed the maximum possible M. But by analysing $x + rx(1 - x/M) > M$, we realized that if there is any $n \geq 1$ for which $r > M/X_n$, then $X_{n+1} > M$, making it impossible for the species to live, and leading to a negative variation and a sudden drop in the biomass of X_{n+2} (the biological species will fight over resources and space). So, convergence is not guaranteed for sufficiently large values of r (3).

The graph of such a case can take many forms - forms describing a growth which we considered uncontrollable - but it is easily visible that the values alternate from being greater than M to being less, as shown in Figures 2, 3, 4 and 5. Figure 5 especially is a good example of what we considered as uncontrollable growth.
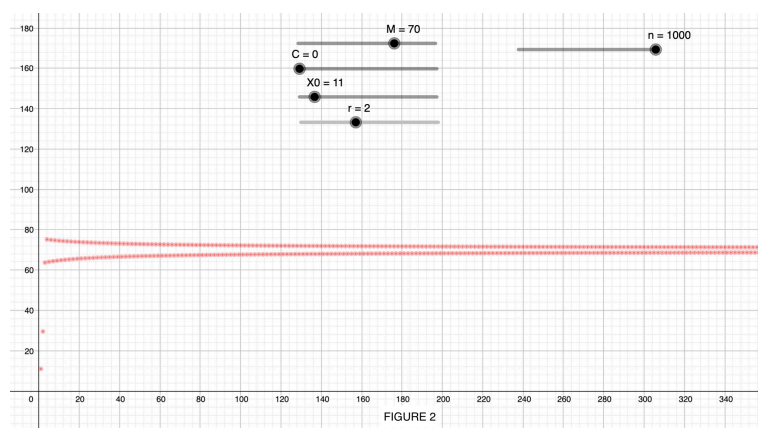
FIGURE 2

Here the rate is 2, the initial biomass is 11 and the maximal one is 70. The complete Geogebra graph shows all values X_n up to $n = 1000$.
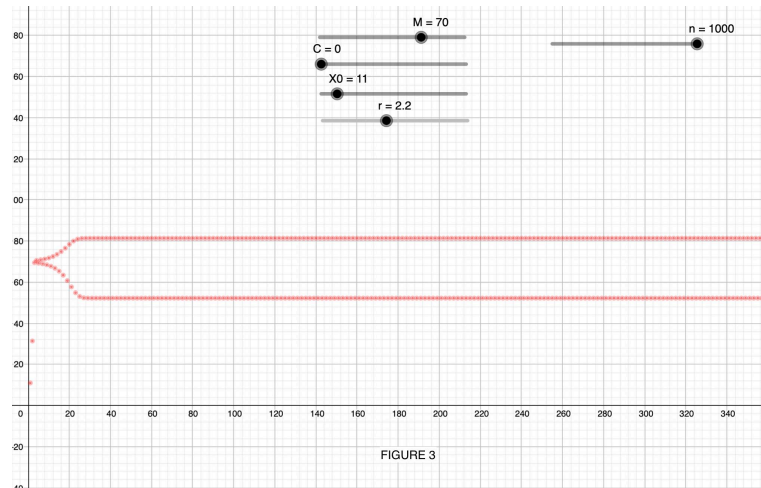
FIGURE 3

In Figure 3, the rate is now 2.2, and the other values are kept the same.
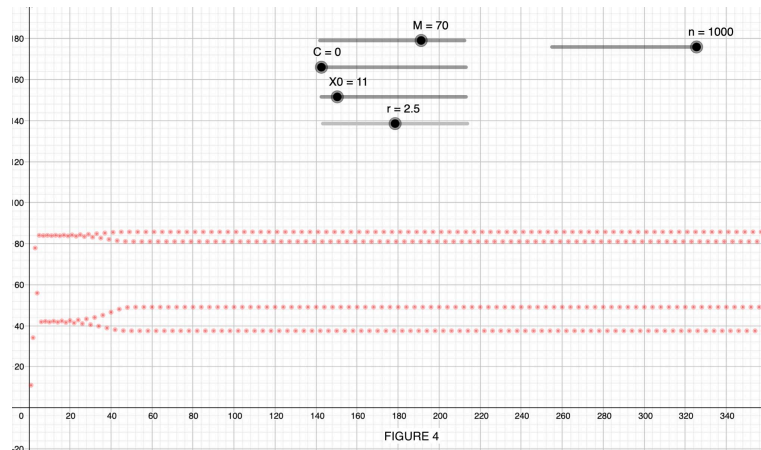

FIGURE 4

Figure 4 presents the same set of parameters for the initial biomass, the maximal biomass and number of generations. But because we want to emphasize what great impact has changing one parameter (namely, the intrinsic growth rate), we set $r = 2.5$.
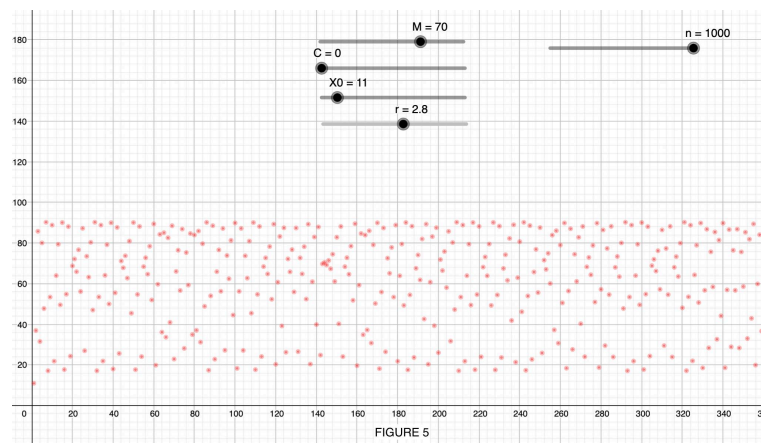
FIGURE 5

In Figure 5, the rate is now 2.8, and the other values are the same. It is easily visible that this growth does not seem to follow a pattern, at least not when looking at a small data set of just 1000 generations (on the figure above, just 360 generations were illustrated, to fit the image format and increase visibility on individual values).

3.1.3 General case: Arbitrary C

This is the case where the fished amount C is an arbitrary nonzero number. Of course, the inequality $C < M$ is true. From our Geogebra graphs and data tests done with the C++ code, we obtained mostly sequences convergent to 0, so cases where the species will go extinct. This time, the variation differential equation (again, a continuous time model is studied) is:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{M}\right) - C,$$

making it harder to integrate if C is arbitrary and not a function of x (4).

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{M}\right) - C \iff \frac{dx}{rx(1 - \frac{x}{M}) - C} = dt \iff \frac{dx}{\frac{rx^2}{M} - rx + C} = -dt$$

$$\iff \frac{M}{r} \cdot \frac{dx}{x^2 - Mx + \frac{CM}{r}} = -dt \iff \frac{M}{r} \cdot \frac{dx}{(x - \frac{M}{2})^2 - \frac{M^2}{4} + \frac{CM}{r}} = -dt$$

Let $u = x - \frac{M}{2}$. This simplifies our calculations, as now we have to integrate

$$\frac{1}{u^2 + M(\frac{C}{r} - \frac{M}{4})}$$

So, we have this new mathematical relation:

$$\frac{M}{r} \int \frac{du}{u^2 + M(\frac{C}{r} - \frac{M}{4})} = -\int dt \iff \frac{M}{r} \int \frac{du}{u^2 + M(\frac{C}{r} - \frac{M}{4})} = -t + \alpha,$$

where α is a constant (found from the initial conditions). How to solve this depends on the sign of $M(\frac{C}{r} - \frac{M}{4})$.

Case 1: $\dfrac{C}{r} = \dfrac{M}{4}$

The new equation is:

$$\frac{M}{r} \int \frac{du}{u^2} = -t + \alpha_1$$

Basic integration rules lead us to the following equivalent equations:

$$-\frac{M}{r} \cdot \frac{1}{u_t} = -t + \alpha_1 \iff u_t = \frac{M}{(t - \alpha_1)r} \iff X_t = \frac{M}{2} + \frac{M}{(t - \alpha_1)r}$$

This integration has to be done only over an interval where the function to be integrated is well defined. Here we must choose $u_t \in (-\infty, 0)$ or $u_t \in (0, +\infty)$, according to $u_0 < 0$ or $u_0 > 0$. In the first case $\alpha_1 > 0$, the solution u_t is not defined for $t \geq \alpha_1$, and we get $u_t \to -\infty$ as $t \to \alpha_1$, which actually means that we get $X_t = 0$ and extinction before $t = \alpha_1$ (5).

The constant should not be independent of X_0, as it is found by analysing the initial biomass value (generation $t = 0$).

$$X_0 = \frac{M}{2} + \frac{M}{-\alpha_1 r} \iff \frac{X_0 r}{M} - \frac{r}{2} = -\frac{1}{\alpha_1}$$

$$\alpha_1 = \frac{2M}{Mr - 2X_0 r}$$

Thus, the value of the biomass of any t-th generation can be determined by the **formula:**

$$X_t = \frac{M\left[t\,r\,(M - 2X_0) - 4X_0\right]}{2\left[t\,r\,(M - 2X_0) - 2M\right]}.$$

Case 2: $\dfrac{C}{r} > \dfrac{M}{4}$

In this case, we can use another notation to simplify the equation. Hence, let N be a $N = \sqrt{M(\frac{C}{r} - \frac{M}{4})}$. That gives us the equivalent equation:

$$\frac{M}{r} \cdot \int \frac{du}{u^2 + N^2} = -t + \alpha_2.$$

When it comes to solving it, we use the calculus knowledge we have.

The equivalence below is valid only when the argument $Nr(\alpha_2 - t)/M$ of the tangent lies between $-\pi/2$ and $+\pi/2$. When $t \to \alpha_2 + M\pi/(2Nr)$, the tangent tends to $-\infty$, which in reality means that extinction occurs before. However, in this case $\frac{du}{dt} \leq -\frac{1}{N^2}$ and this clearly shows that we have extinction, without the need of calculations.

$$\frac{M}{r}\frac{1}{N} \cdot \arctan\left(\frac{u_t}{N}\right) = -t + \alpha_2 \iff \frac{u_t}{N} = \tan\left(\frac{Nr(-t + \alpha_2)}{M}\right)$$

$$\iff u_t = \sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)} \cdot \tan\left(\frac{r \cdot \sqrt{M(\frac{C}{r} - \frac{M}{4})}(-t + \alpha_2)}{M}\right)$$

$$\iff X_t = \frac{M}{2} + \sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)} \cdot \tan\left(\frac{r \cdot \sqrt{M(\frac{C}{r} - \frac{M}{4})}(-t + \alpha_2)}{M}\right),$$

where α_2 can be found from the initial data, i.e. from X_0.

$$\alpha_2 = \frac{\sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)} \cdot \arctan\left(\frac{(2X_0 - M) \cdot \sqrt{M(\frac{C}{r} - \frac{M}{4})}}{2M \cdot (\frac{C}{r} - \frac{M}{4})}\right)}{r \cdot \left(\frac{C}{r} - \frac{M}{4}\right)}.$$

Putting together these last two formulas, we obtain that, at the t-th generation, the biomass value will be (6):

$$X_t = \frac{M}{2} + \sqrt{M\left(\frac{C}{r} - \frac{M}{4}\right)} \cdot \tan\left(\frac{r \cdot \sqrt{M(\frac{C}{r} - \frac{M}{4})}(-t + \frac{\sqrt{M(\frac{C}{r} - \frac{M}{4})} \cdot \arctan(\frac{(2X_0 - M) \cdot \sqrt{M(\frac{C}{r} - \frac{M}{4})}}{2M \cdot (\frac{C}{r} - \frac{M}{4})})}{r \cdot (\frac{C}{r} - \frac{M}{4})})}{M}\right).$$

Case 3: $\dfrac{C}{r} < \dfrac{M}{4}$

In this particular case, $M(\frac{C}{r} - \frac{M}{4}) < 0$, making us define the helpful parameter N as $N = \sqrt{M(\frac{M}{4} - \frac{C}{r})}$. The equation now looks like the following:

$$\frac{M}{r} \cdot \int \frac{du}{u^2 - N^2} = -t + \alpha_3.$$

Solving this requires much more advanced knowledge. It is almost basic calculus until we obtain the relation below:

$$\ln\left|\frac{u_t - N}{u_t + N}\right| = \frac{2Nr(-t + \alpha_3)}{M} \iff \left|\frac{u_t - N}{u_t + N}\right| = e^{\frac{2Nr(-t + \alpha_3)}{M}}.$$

This case we thought was uncontrollable, because we couldn't at first analyse whether u_t is greater or smaller than N. But again, integration has to be done on an interval where

---

$(u_t - N)/(u_t + N)$ has constant sign, i.e. $(-\infty, N), (-N, N)$ or $(N, +\infty)$ depending on the initial value and there is no jump from one of these intervals to another (this might be different in the discrete time model).

Extinction may still occur. Namely, if $u_0 < -N$, we have $(u_0 - N)/(u_0 + N) > 1$ and thus $\alpha_3 > 0$. Then we get

$$u_t = N \frac{e^{2Nr(\alpha_3 - t)/M} + 1}{e^{2Nr(\alpha_3 - t)/M} - 1}$$

as long as this function is continuous, i.e. for $t < \alpha_3$, and when $t \to \alpha_3$, then $u_t \to -\infty$ (7).

3.1.4 General case: $C = p\,X_n$

This is the case where the fished quantity is calculated every year, as a part p of that generation's biomass, meaning that $C = p \cdot X_n$. Moreover, this is most likely to be the situation closest to reality, as it is the most reasonable when it comes to choosing the fishing biomass with regards to the current quantities.

The recurrence relation here is:

$$X_{n+1} = X_n + rX_n \left(1 - \frac{X_n}{M}\right) - pX_n$$
$$= X_n + rX_n \left(\frac{r - p}{r} - \frac{X_n}{M}\right)$$

The steps we are going to follow are the same as in subsection 3.1.2. Let's study the the variation and possible equilibrium points. Defining the variation between generation $t + 1$ and t in the same way as in the previous subsections, we are able to state that $\Delta x = \frac{dx}{dt} = rx\left(\frac{r-p}{r} - \frac{x}{M}\right)$.

The equilibrium points are the positive real roots of $\Delta x = 0$. Hence, 0 and $\frac{M(r-p)}{r}$ are the values we were looking for.

Now, as for the sign of Δx – sign which determines whether the sequence is ascending or descending –, we can easily see that it is all a matter of comparing X_t to $M(r-p)/r$. While $X_t < M(r-p)/r$, then the sequence is ascending. Else, while $X_t > M(r-p)/r$, then the sequence is descending. But because $M(r-p)/r$ is an equilibrium point and supposing X_0 is non-null, then X_t should converge to $M(r-p)/r$ in the continuous time model. In the discrete time model, though, periodic or chaotic behavior is possible, as in subsection 3.1.2. (which is the particular case $p = 0$).

We wanted to find the formula for any X_t, by solving the differential equation:

$$\frac{dx}{dt} = rx\left(\frac{r - p}{r} - \frac{x}{M}\right).$$

We will arrive to a logarithmic equation involving a modulus and just as we cannot state the monotony of the sequence because we must compare X_t to $\frac{M \cdot (r-p)}{r}$, now we cannot decide the sign of the expression within the modulus.

3.2 Second Approach

This approach belongs to Petru Săveanu and Hugo Giret.

We took the functions g and f where:

$$g(x) = x\,(r+1) - r\,\frac{x^2}{M} - C \quad \text{and} \quad f(x) = r\,x - r\,\frac{x^2}{M} - C,$$

$(g(x) = x + f(x))$.

In other words, the function f is the difference between 2 years and the discriminant of the quadratic polynomial f is:

$$\Delta = r^2 - 4\,r\,\frac{C}{M}$$

We chose to study the function f because it explains the evolution from year to year. For example, if the function is positive, it means an increase in quantity, if it is negative it will lead to extinction, and if it is once zero, the result will be stagnation.

3.2.1 Case 1: $\Delta < 0$

For all x, $f(x) < 0$, so $g(x) = x + f(x) < x$. Then $X_0 > X_1 > \ldots > X_n$.

FIGURE 6
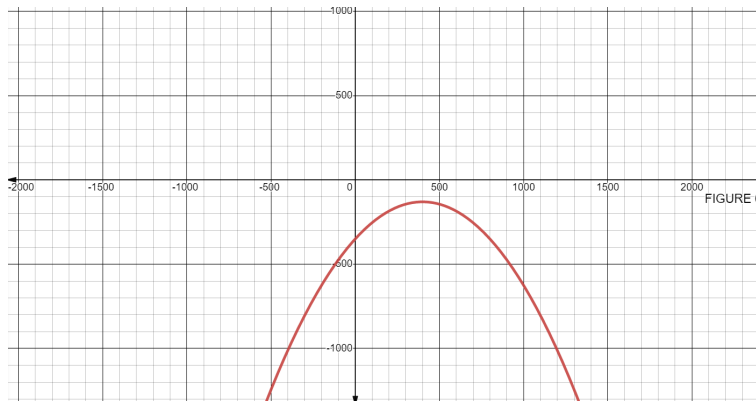
Considering the graph of the function f, the differences will get bigger and bigger, by approaching the quantity of 0, the origin of the system, and the differences are bounded from below, so the result will be obvious extinction.

3.2.2 Case 2: $\Delta = 0$

$$\frac{C}{r} = \frac{M}{4} \Rightarrow f\left(\frac{M}{2}\right) = r\,\frac{M}{2}\left(1 - \frac{\frac{M}{2}}{M}\right) - C = \frac{r\,M}{4} - C = 0$$

The only point where $f(x) = 0$ is $\frac{M}{2}$ (and $f(x) < 0$ elsewhere). If X_0 is less than $\frac{M}{2}$ the result will be extinction, because we are in the same situation as in the previous case.
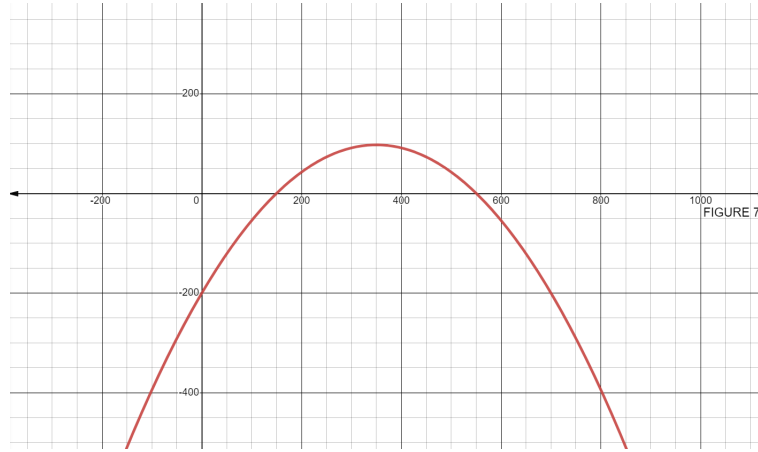
Thus, we will look for the points that bring the biomass below the value of $\frac{M}{2}$, trying to calculate the distance from $\frac{M}{2}$, called a (8).

$$g\left(\frac{M}{2} + a\right) < \frac{M}{2} \iff \frac{M}{2} + a + r\left(\frac{M}{2} + a\right)\left(1 - \frac{\frac{M}{2} + a}{M}\right) - \frac{Mr}{4} < \frac{M}{2}$$

$$\iff a < a^2\,\frac{r}{M} \iff \frac{M}{r} < a$$

This means that if X_0 belongs to the interval $\left(\frac{M}{2}, \frac{M}{2} + \frac{M}{r}\right)$, $g(x) > \frac{M}{2}$, but because $f(x) \le 0$ for all x, $X_0 > X_1 > \ldots$, the decrease limit being $\frac{M}{2}$.

If X_0 does not belong to that range, the result will be extinction.

3.2.3 Case 3: $0 < \Delta \leq 1$

FIGURE 7

We chose this case because we can choose a clearly defined interval to obtain balance, and uncontrolled growth is impossible. We used the graph of the function f and the first derivative of the function g to obtain intervals of monotony:

$$g'(x) = r + 1 - x\,\frac{2\,r}{M}$$

For any x in range of $x_1 = \frac{r - \sqrt{\Delta}}{2r/M}$ and $x_2 = \frac{r + \sqrt{\Delta}}{2r/M}$, where $f(x_1) = 0$ and $f(x_2) = 0$, we have $g'(x) \geq 0$ since

$$g'(x_2) = r + 1 - \left(\frac{r + \sqrt{\Delta}}{\frac{2\,r}{M}}\right)\frac{2\,r}{M} = r + 1 - r - \sqrt{\Delta} = 1 - \sqrt{\Delta}$$

$$x_1 < x < x_2 \Rightarrow g'(x_1) > g'(x) > g'(x_2) = 1 - \sqrt{\Delta} \geq 1 - 1 = 0$$

So, $x_1 = g(x_1) < g(x) < g(x_2) = x_2$ if $x_1 < x < x_2$.

Using $f(x) > 0$ when $x_1 < x < x_2$, we obtain $x_1 < X_0 < X_1 < X_2 < \ldots < X_n < x_2$, balance, if $x_1 < X_0 < x_2$, and $x_1 > X_0 > X_1 > X_2 > \ldots > X_n$ meaning extinction if $X_0 < x_1$.

For $X_0 > x_2$ we can obtain balance or extinction, depending on the distance from x_2 (9).

3.2.4 Case 4: $\Delta > 1$

This is the only situation when X_n could be greater than M. Due to the very large variation of values in two consecutive years, we studied the situation when X_1 is greater than M.

We took the function h, where $h(x) = g(x) - M$ and we calculated the roots of the equation.

$$h(x) = g(x) - M = x\,(r + 1) - r\,\frac{x^2}{M} - C - M \Rightarrow \Delta' = (r + 1)^2 - 4r\,\frac{C + M}{M}$$

Then we consider the points $P = \frac{r + 1 - \sqrt{\Delta'}}{2r/M}$ and $W = \frac{r + 1 + \sqrt{\Delta'}}{2r/M}$ for which $g(P) = M$ and $g(W) = M$ (10).

Because X_0 is smaller than M, and $g(x) > x$ only if $x_1 < x < x_2$, we create an interval $\big(\max(x_1, P), \min(x_2, W)\big)$ by joining the interval (x_1, x_2) with the interval (P, W) (11). If X_0 belongs to that interval, $X_1 \geq M$.

Using the properties of the functions f and g, which depend on the values of the discriminant, we can manipulate C, r and X_0 to obtain extinction, equilibrium and uncontrolled growth.

4 Conclusions

Looking at the first approach, we can observe that formulas can be found, but they are different and depend on the relationship between the given parameters C, M and r. Moreover, some cases are harder to follow and, from the knowledge we possess now, can be considered as cases of uncontrollable growth.

From our previous work, we observed different types of curves depending on the value of $\Delta = r^2 - 4r\frac{C}{M}$: The extinction curve for $\Delta < 0$, the constant curve, the "alternating"/"sinusoidal" curve, the scattered one, etc.

We can introduce an environmental variable considered as a random variable V_n that can take for example the three possible values 0.5, 1, and 1.5, which represent the variability of the year n, and it is introduced the model: $X_{n+1} = (X_n + r \cdot X_n \cdot (1 - \frac{X_n}{M}) - C) \cdot V_n$; we obtain unusual cases, creating new types of curves that seems non-periodic and hard to define. Studying this evolution type is one of our next research goals.

The Schaefer Model is not a perfect one: it is a discrete mathematical model, meaning that the fish population is calculated every year; in real life, environmental facts and events could influence so much that the fish could disappear in 6 months, for example, and so the model would not work. To be more accurate, we should take more parameters (related to the aquatic habitat) into consideration.

5 Other Contributors

Other contributors to this research topic within the "Mathematics for Sustainable Development" project were Codruța Jucan, Horațiu Cătărig, Alexandru Coroiu, Sony Șipos, Dimitri Czornomaz, Alexis Dekeyser, Théo Brittain, Diégo Mekarcha, Lou-Anne Le Bris, Lucas Fernandes, Valentin Virga.

6 References

1. C-H. Wang and S-B. Wang, *Theoretical development of Schaefer model and its application*, Kolonia, Pohnpei State, Federated States of Micronesia: Western and Central Pacific Fisheries Commission. Meeting of the Scientific Committee of the Western and Central Pacific Fisheries Commission, Noumea, New Caledonia, 8-19 August 2005, 1st. 16 p.

2. J. Zhang and M-D. Smith, *Estimating a Generalized Gordon-Schaefer Model with Heterogeneous Fishing Data*, in: Proceedings of the Thirteenth Biennial Conference of the International Institute of Fisheries Economics & Trade, July 11-14, 2006, Portsmouth, UK *Rebuilding Fisheries in an Uncertain Environment*. Compiled by Ann L. Shriver. International Institute of Fisheries Economics & Trade, Corvallis, Oregon, USA, 2006. CD ROM. ISBN 0-9763432-3-1

3. J-C. Robinson, *An Introduction to Ordinary Differential Equations*, Cambridge University Press, 2004, The Edinburgh Building, Cambridge CB2 2RU, UK. Published in the United States of America by Cambridge University Press, New York, ISBN-10 0-511-164831-1

Editing Notes

(1) This recurrence relation is closely related to the so-called *logistic map* $\left(x_{n+1} = ax_n(1-x_n)\right)$.

(2) In this program, it is assumed that extinction occurs, which is of course not always the case. The loop "while(x[i]>0)…" should also be stopped before the table size is reached.

(3) More precisely, we have $x + rx(1 - x/M) > M$ when x is between M/r and M. So, starting with $X_0 \in [0, M]$, when $r \leq 1$ we shall get $X_n \in [0, M]$ for all n, an increasing sequence and convergence towards M, as explained above. But as soon as $r > 1$, the same proof shows that the sequence is increasing as long as $X_n < M/r$ so that for some n we get $X_n > M/r$ and then $X_{n+1} > M$. However convergence can still occur if r is not too large.
A wide variety of behaviors can be found by increasing the parameter r. In the figures below, we first see convergence, then oscillation between two values, then between four values and then chaos in the last case.

(4) In the next equations, before dividing by $rx\left(1 - \frac{x}{M}\right) - C$ it should be mentioned that the roots of this polynomial, denoted by x_1 and x_2 in section 3.2.3 below, yield constant solutions $X_t = x_1$ and $X_t = x_2$, and that in any other case we can assume that the denominator is non-zero.

(5) So, we have extinction if $u_0 = X_0 - \frac{M}{2} < 0$, i.e. if $X_0 < \frac{M}{2}$, and then the formula for X_t given below is valid only for $t < \alpha_1$. The case $X_0 = \frac{M}{2}$ yields the constant solution $X_t = x_1 = x_2 = \frac{M}{2}$ and in the case $X_0 > \frac{M}{2}$, we have $\alpha_1 < 0$, the given solution X_t is well defined for all $t \geq 0$ and converges towards $\frac{M}{2}$.

(6) Again, the formula below is valid only before extinction.

(7) Here, $u_0 = N$ and $u_0 = -N$ yield the constant solutions, and in the cases $-N < u_0 < N$ or $u_0 > N$ the same calculation yields solutions u_t well defined for all $t > 0$ and converging towards N.

(8) Here, a is assumed to be positive.

(9) As mentioned by the authors in a previous version of this work, we can exploit the symmetry of the function g: we have $g\left(\frac{r+1}{r} M - x\right) = g(x)$; so, when $\frac{r+1}{r} M - x_2 \leq X_0 < \frac{r+1}{r} M - x_1$ then $x_1 < X_1 \leq x_2$ and we obtain convergence towards x_2, and when $X_0 > \frac{r+1}{r} M - x_1$ then $X_1 < x_1$ and there is extinction.
In the remaining case $x_2 < X_0 < \frac{r+1}{r} M - x_2$, one can show $x_2 < \ldots < X_1 < X_0$ and again convergence towards x_2 (the proof is left to the reader).

(10) These points P and W may not exist, since the discriminant Δ' can be negative, depending on the values of r and C/M, and then $g(x) < M$ for all x.

(11) We have $x_2 \leq M$ since $\Delta \leq r^2$ and $g'(x) < 0$ for $x \geq x_2$ since $\Delta > 1$, from the above proof section 3.2.3. It follows that we cannot have $g(x) > M$ outside (x_1, x_2) and the new interval $\left(\max(x_1, P), \min(x_2, W)\right)$ is simply (P, W).
However, the fact that X_0 belongs to this interval is not enough to determine if we have extinction, convergence or "uncontrolled growth". This case includes the particular case $C = 0$, $r > 1$, studied section 3.1.2 (if $C = 0$, then $\Delta = r^2$) where different evolutions appear, but the problem of deciding which kind of evolution occurs according to the parameter value remains unsolved