Inflated sets

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I. Introducing the research topic

We take a convex plane figure. The diameter of the figure is the greatest distance between two points of this figure. To this figure (A) we can "add" a point M outside the figure by considering the convex envelope formed by A and M. It is like placing a rubber band around A and M.
A figure is said to be "inflated" when the addition of any point in the plane (in the above sense) increases its diameter.
Try inflating a square and other shapes. What can be said about inflated figures of the same diameter?
We call a closed planar curve an ’inflated set’ when the addition of any point increases its diameter (the longest distance between any two points of the figure). The diameter of a subset of a metric space is the least upper bound of the set of all distances between pairs of points in the subset. Our approach has been to try to find a handful of general properties and theorems that could improve our understanding

of inflated sets and that could spark further research topics. This inflation theme essentially proposes that we modify the figure, according to d, the maximum distance from a point of the figure to another point of the figure, so that each and every point of the figure has a corresponding point on the outer surface of the figure, situated at the distance d of it. Thus, adding any other exterior points will increase the distance d, the figure therefore needing further inflation.
Moreover, this work is concerned with points and figures in the Euclidean plane. In some sections it turns out to be useful to fix an orthogonal system of coordinates and identify the Euclidean plan with R^2.
Computing the diameter of a set of points in the Euclidean plane is a fundamental problem in computational geometry, which is presented in Shamos and Hoey's *Closest-point problems*[2].
In the proofs of the results presented in the article, we make use of foundational concepts from analytic geometry. For a better understanding and detailed exposition of these concepts, readers may refer to Douglas Riddle's dedicated work on the subject, *Analytic Geometry*[1].

II. Methods and results

1. Inflating figures

1.1. Inflating a point

Let F $\subset R^2$ be a convex planar figure and let d be its diameter. The process of inflation is an iterative one, in which, at first step, we add to the figure a M in the plane such that $d(M, P)\leq d$, for all $P \in F$ and $d(M,P)=d$. This step is repeated and ends when one obtains an inflated set. (1) The locus of the points in a plane equally distanced from a point *P* is a circle - the inflated figure of a segment shall be the circle with the diameter being the segment previously inflated. Taking into account the conditions imposed by the process of inflating figures, whatever point in the plane we take will be situated on the circle since the maximum distance is the diameter of that circle.

1.2. Inflating a circle/circular shapes

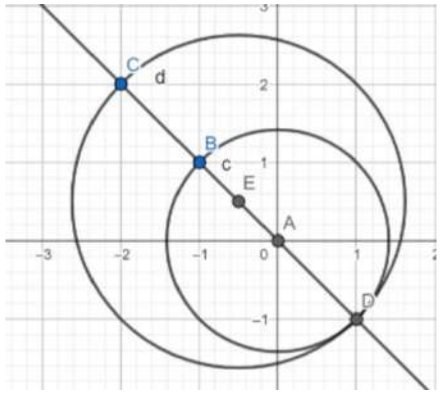
1.2.1 Inflating a disk

Inflating a disk - any disk is already an inflated figure. We can further inflate a disk (an already inflated figure) by tracing a bigger circle around it, which engulfs it completely - that would be another inflated figure.

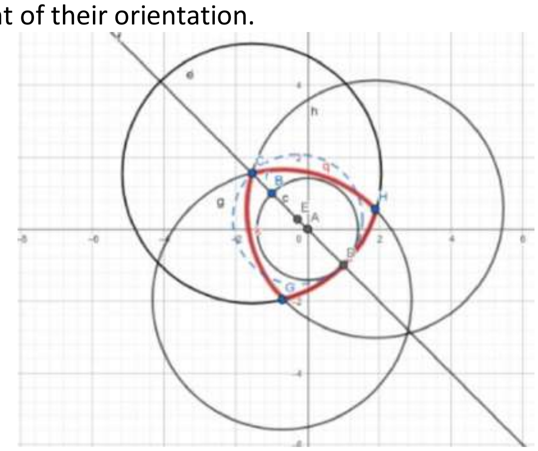
1.2.2 Inflating a circle and an exterior point (2)

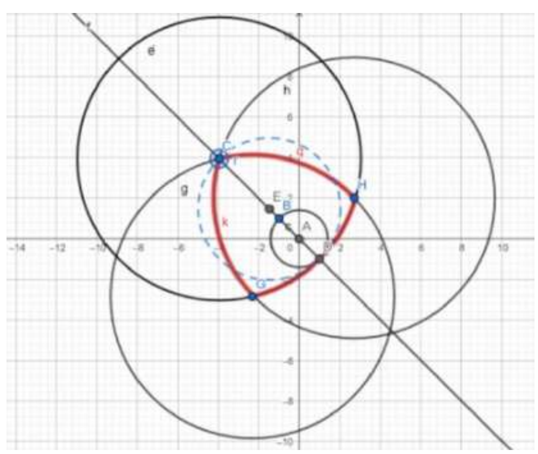
Let's say we have a circle $C(A,r)$ and an exterior point C. We will write $C(A, r)$ for the circle of center A and radius r. Let us inflate the figure $F=C \cup \{C\}$.
For inflating the figure, we trace a straight line containing the circle's center, *CA,* and denote *B* and *D* the intersections of the line with the circle *C, D* being the furthest point from *C;* we trace the midpoint of *CD* of the maximum-length segment (the point and the point on the circle furthest of it - $CD = d$) and then we draw the circle with the center in that midpoint *E* and of radius $\frac{d}{2}$. The diameter of the figure is the length $d=CD$. (3)

1. Riddle,Douglas F. , *Analytic Geometry*, Cengage Learning; 6th edition (October 25, 1995)
2. Shamos, M.I. & Hoey , D. (1975).*Closest-point problems.* In Proceedings of the 16th annual Symposium on Foundations of Computer Science (pp. 151-162)



We can also inflate this particular figure by constructing its corresponding Reuleaux triangle, which is a curved triangle with constant width, the simplest and best known curve of constant width other than the circle. It is formed from the intersection of three circular disks, each having its center on the boundary of the other two. Constant width means that the separation of every two parallel supporting lines is the same, independent of their orientation.

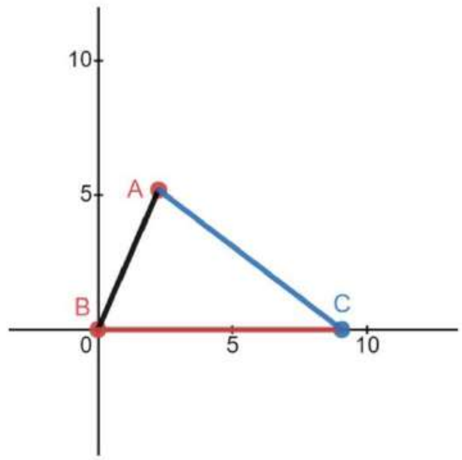




Observation: it does not matter where the circle’s center and the exterior points are placed, because the obtained Reuleaux triangle completely engulfs the initial circle and the initial exterior point.

1.3. Inflating a triangle

Let us take the scalene triangle with coordinates $A(a_1, a_2)$, $B(0,0)$ and $C(c_1, 0)$, in the xBy coordinate system.



The equations of the sides of the triangle are:

$$AB: y = \frac{x \cdot a_2}{a_1}, 0 \le y \le a_2$$
$$BC: y = 0, 0 \le x \le c_1$$
$$AC: y = \frac{x \cdot a_2 - c_1 \cdot a_2}{a_1 - c_1}, 0 \le y \le a_2$$

These were obtained by applying the line equation formulas to the points A, B and C of ΔABC, and then setting conditions in order for them to be only segments, not lines.

For inflating the triangle, we must first determine its diameter. We know that the diameter must be between two vertices of the triangle. So, in essence, what we need to do first is calculate the lengths of the sides of the triangle and then compare them one with another, to determine the diameter. (4)

Let us take $AB = c$, $AC = b$ and $BC = a$. Therefore,

$$a = c_1$$
$$b = \sqrt{(a_1 - c_1)^2 + a_2^2}$$
$$c = \sqrt{a_1^2 + a_2^2}$$

Let d be the diameter of this figure. We shall now have three distinct cases:

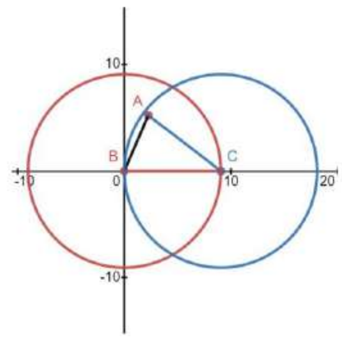
1. $AB = d$
2. $AC = d$
3. $BC = d$

The easiest case to treat is $BC = d$, equivalent to $a = d$ or, in terms of distance from the origin of the system the (point B), $c_1 = d$.
To build the inflated figure, first we need to trace the two circles from the vertices B and C of the longest side of the triangle. The circle with the center in $B(0,0)$ and the radius d shall have the equation

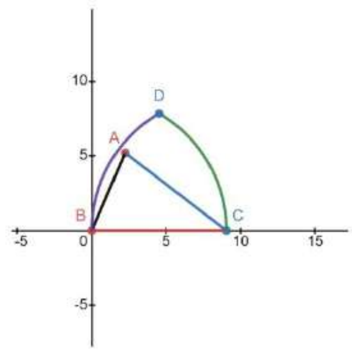
$$x^2 + y^2 = d^2 \qquad (1)$$

and the circle with the center in $C\ (c_1, 0)$ and the radius d shall have the equation

$$(x - d)^2 + y^2 = d^2 \quad (2)$$



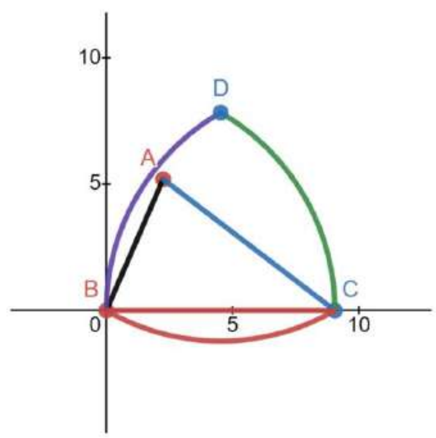
By intersecting these two circles, in the semiplane $y > 0$, we shall obtain point D, of coordinates $D(\frac{d}{2}, \frac{d\sqrt{3}}{2})$.



We must take only the arcs of these two circles that are from C to D and from D to B respectively, so for equation 1 we shall have that $\frac{d}{2} \leq x$, which is D's x-coordinate, and $y > 0$, which is C's y-coordinate, and for equation 2 we shall have that $x \leq \frac{d}{2}$, which is, again, D's x-coordinate, and $y > 0$, being B's y-coordinate.
The last step in inflating this triangle is tracing the circle with the centre in D and the radius d, taking only the arc from B to C. The equation of this circle shall then be

$$(x - \frac{d}{2})^2 + (y - \frac{d\sqrt{3}}{2})^2 = d^2 \quad (3)$$

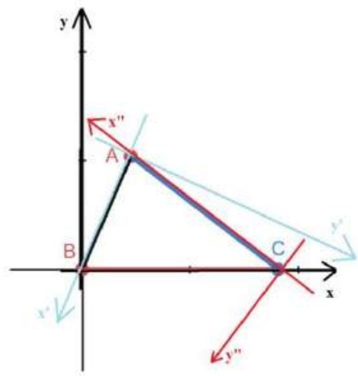
and its restrictions shall be $0 \leq x \leq c_1$ and $y \leq 0$.



Whatever we move the points A and C so that BC is of maximum length among the triangle's sides, the figure will stay the same.
We can generalise for the other two cases (because the calculations are tedious and repetitive), by seeing which is the longest side and then moving the coordinate system so that the x-axis contains it, with an apex of that side is its origin. Applying this for fixed points A, and then C - moving (translation + rotation) the coordinate system's center gives us the following results:

$$\begin{cases} xBc : \Delta ABC \\ x'Ay' : \Delta CAB \\ x"Cy" : \Delta BCA \end{cases} \tag{4}$$

For xBc, the initial A goes to C, the initial B to A and the initial C to B, and for xCy, the initial A goes to B, the initial B to C and the initial C to A, these being essentially permutation cycles.

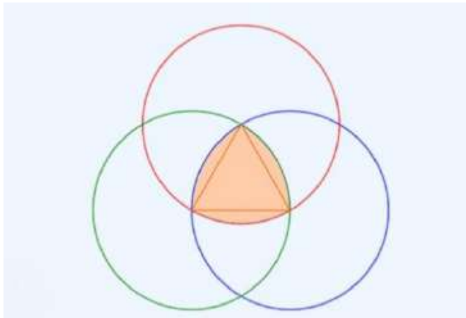


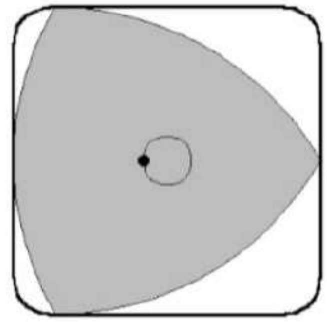
The xBc system of coordinates is in black, the xAy system is in light blue and the xCy one is in red.

Observation: We can, more easily, inflate a scalene triangle by taking its longest side, d, making it an equilateral triangle (we shall leave that side d untouched, only focusing on the other ones), and then inflating it so that it becomes a Reuleaux triangle. Every triangle can be, thus, reduced to a Reuleaux one, and the positioning of point D, at coordinates $(\frac{l}{2}, \frac{l\sqrt{3}}{2})$, only further approves of this fact - in an equilateral triangle ΔABC of side d, the altitude is exactly $\frac{d\sqrt{3}}{2}$. Thus, for easily inflating a triangle, we can make it its corresponding Reuleaux triangle. This also works for obtuse-angled triangles, since the biggest side, d, will be the one opposing the obtuse angle (from Pythagora's generalised theorem), thus the corresponding equilateral triangle will engulf all of the initial triangle.

1.3.1. The Reuleaux Triangle (5)

When inflating an equilateral triangle $\triangle ABC$, we arrive at a special shape called the Reuleaux Triangle, defined by the intersection of the three circular discs $D(A, l)$, $D(B, l)$, $D(C, l)$, where l is the length of each side of $\triangle ABC$.
Its most fundamental property is that it has constant width (meaning that it has the same width in every direction). Not only that, but it is also the shape of the smallest area possible $(\frac{l^2(\pi-\sqrt{3}}{2})$ having this property. Moreover, the Reuleaux triangle can rotate fully within a square, which gives it a compact design with multiple technical applications.



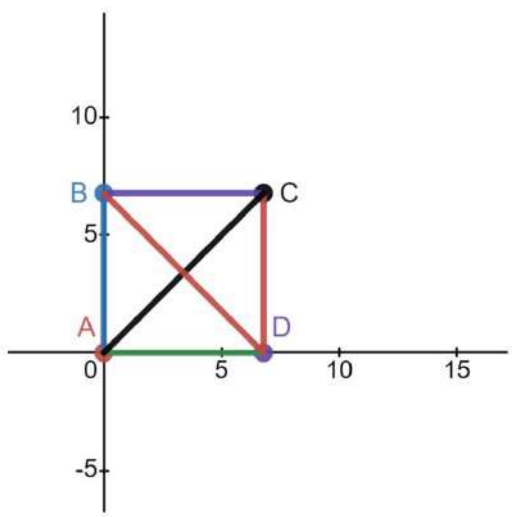


1.4. Inflating a square

Our initial hypothesis was that we can inflate the square by making it its circumscribed circle, as you can see in this figure, but from what we can deduce, the area of the figure must be minimal, hence the unusual shape given in the initial problem was chosen for the inflation.

If we were to take a square with its side of length l, the area of its circumscribed circle would be $\pi \times \frac{l^2}{2}$. Let us take the square $ABCD$ of side l with coordinates $A(0,0)$, $B(0,l)$, $C(l,l)$ and $D(l,0)$. The diameter of the square is the maximum distance between two points on its contour, so that must be the length of its diagonal, $d = l\sqrt{2}$. The equations for the sides of the square are:

$$AB : x = 0, 0 \leq y \leq l$$
$$AD : y = 0, 0 \leq x \leq l$$
$$BC : y = l, 0 \leq x \leq l$$
$$CD : x = l, 0 \leq y \leq l$$



We then take, let's say, AC as the diameter. We shall construct the two circles of radius d and centres A and D, respectively. Their equations will be $x^2 + y^2 = d^2$, which can be written as

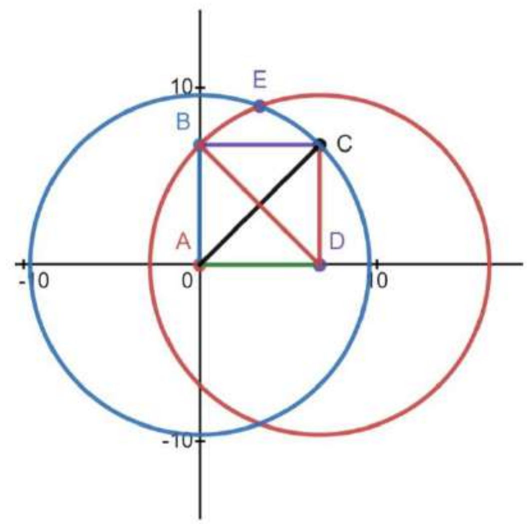
$$x^2 + y^2 = 2 \times l^2,\ x \geq 0,\ l \leq y \leq \frac{l\sqrt{7}}{2} \qquad (5)$$

for circle $C(A,d)$, with the due restrictions, and $(x-l)^2+y^2=d^2$, which can be written as

$$(x-l)^2+y^2=2\times l^2,\ 0\leq x\leq \frac{l}{2}, y\geq 0 \qquad (6)$$

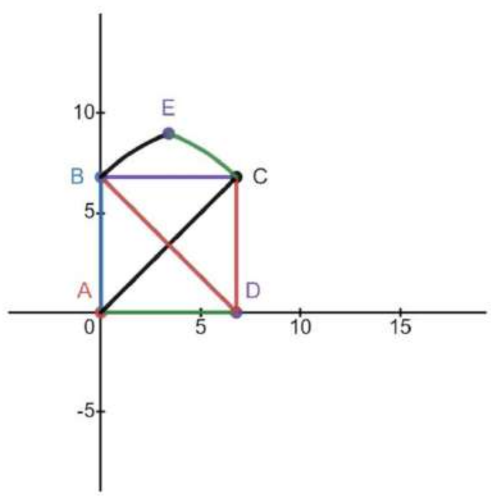
for circle $C(D,d)$, with the due restrictions (they will come in handy for point F, obtained in the exact same way as point E, only from circles traced with the centres in A and B, this time, to cover the whole figure).

By intersecting them and taking the point in the same semiplane fenced by AD as the rest of the square, we find point E, located at distance d from both A and D. Thus, E is on the perpendicular bisector of the $[AD]$ segment, and we can conclude that its x-coordinates are the mean between the x-coordinates of A and of D, which would give us $x_E=\frac{l}{2}$. The figure without restrictions is, thus:



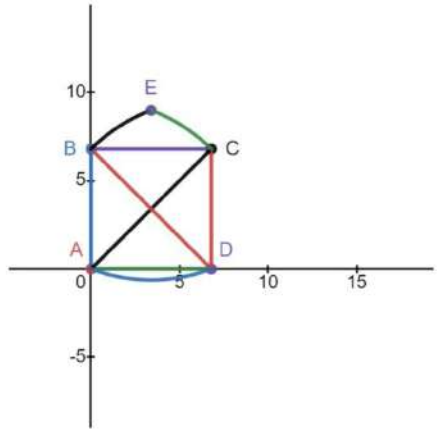
Knowing its x-coordinates and the coordinates of the circles for which it represents a point of intersection, we can put that x-coordinate into one of the equations above, finding $y_E=\frac{l\sqrt{7}}{2}$, because it needs to be a positive number, being above the x-axis of coordinates.

Now, the restrictions from equations 5 and 6 hopefully make a bit more sense, giving us the following figure:



Now we shall trace arc AD from the circle of centre E and radius d, so that E can have its corresponding points of distance d from it, taking from circle $C(E,d)$ only the AD arc, of equation

$$(x-\frac{l}{2})^2+(y-\frac{l\sqrt{7}}{2})^2=2\times l^2, y\leq 0$$



We shall now repeat these steps for A and B, noticing that we can continue the process of inflation, of adding points so that each and every point on the contour has its correspondent point at distance d from it, by exactly rotating this figure 90°.
We shall take the circles $C(A,d)$ (see equation 5) and $C(B,d)$, of equation

$$x^2+\ (y-l)^2=2\times l^2, \qquad 0\leq x, 0\leq y\leq \frac{l}{2}$$

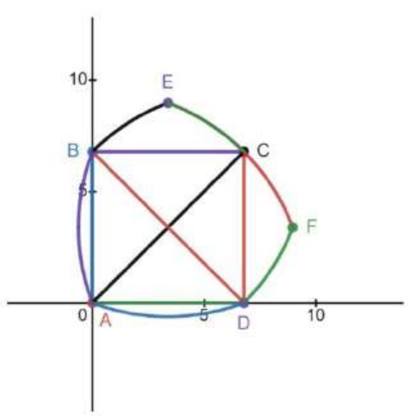
We can see that point F is thus equally distanced from both A and B, this placing it on $[AB]$'s perpendicular bisector and giving it the y-coordinate of $\frac{l}{2}$. By putting this coordinate into one of the equations for $C(A,d)$ or $C(B,d)$ and by taking only the positive solution, we obtain $(\frac{l\sqrt{7}}{2},\frac{l}{2})$.

Observation: equation 5 has now the restrictions $\frac{l}{2}\leq y\leq\frac{l\sqrt{7}}{2}$, now that we see that both E and F are on the same circle - $C(A,d)$.

We now proceed to trace arc AB of the circle $C(F,d)$:

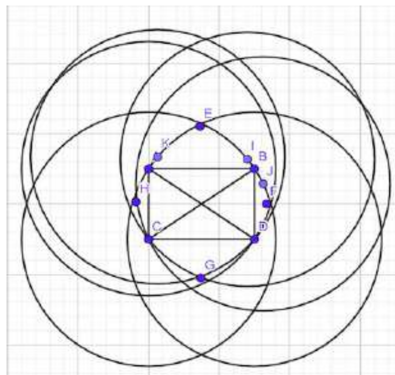
$$(x-\frac{l\sqrt{7}}{2})^2+(y-\frac{1}{2})^2=2\times l^2, 0\geq y$$

We have now obtained the full inflated figure for a square, and we can now modify the length of the side l however we want, because the equations will yield the same results - the following figure: (6)



1.5. Inflating a rectangle

The rectangle's diameter d is equal to its diagonal, and from Pythagora's theorem, we can deduce that $d = \sqrt{l^2 + L^2}$, where l and L are the lengths of its sides. The process is similar to that of inflating a square.



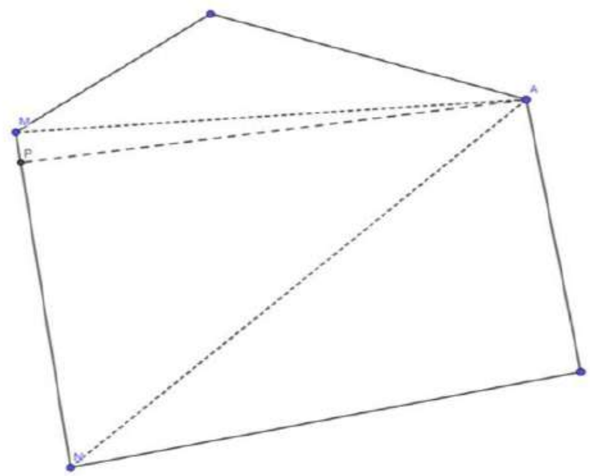
2. Properties of Inflated Sets

2.1 The Diameter

Let us prove that the diameter (the longest segment encapsulated within the figure) of any convex polygon is a segment that unites two vertices of the polygon. Firstly, let us make use of the fact that the diameter is achieved by two points on the sides of the polygon. The proof can be reached intuitively. If, on the contrary, the segment has one or two extremities situated in the interior of the polygon, it can be extended along its analogous line until in intersects the corresponding side of the polygon, thus obtaining a longer segment (so the shorter one could not have been the diameter). Afterwards, let us prove that the diameter is determined by two vertices of the polygon. Let there be a point A on the outline of the figure and an edge $[MN]$. If we arbitrarily choose a point $P \in [MN]$ so that $AP \perp MN$, one of the angles $< APM$ and $< APN$ will be obtuse, and the other will be acute (as they together form 180). Should we assume that $< APN$ is the obtuse angle, it will be the widest angle in $\triangle APN$. Opposite the widest angle of the triangle lays the largest side, therefore $> AP$. Similarly, should $< AMP$ be the obtuse angle, we obtain $> AP$, and should we choose the point $P \in [MN]$ so that $AP \perp MN$, $[AM]$ and $[AN]$ would be the hypotenuses of the right-angled triangles $\triangle APM$ and $\triangle APN$, thus $AM > AP$ and $AN > AP$. In conclusion, there will always be a segment determined by two vertices longer than any segment determined by points on the polygon (excluding the vertices). Consequence: For the polygon $A_1 A_2 \dots A_n$ we calculate the diameter $d = max(A_i A_j)$, $i, j \in 1,2, \dots, n$, and draw the circular discs $D(A_1, d)$, $D(A_2, d)$, ...,$D(A_n, d)$. The diameter of the polygon will increase if and only if we add a point outside of the intersection of the circles. In other words, if we add a point P to the polygon, the change in diameter $\Delta d = 0 \leftrightarrow P \in \cap_{k=1}^{n} D(A_k, d)$. This is the code for a C++ program that takes the coordinates of the polygon and k additional points as inputs, and ottputs the diameter, and whether adding each of the k points would change the diameter.

2.2 The Outline

The outline of an inflated set is a collection of lines that are necessarily arcs of circles, and never straight lines. The rationale used to prove this is similar to the one used in 2.1. Should there be a segment $[MN]$ on the outline of an inflated set, we can take a vertex A and a point $P \in (MN)$. However, AP is larger than either AM or AN, or both. Therefore, we can choose a point $Q \in [AP \setminus [AP]$ so that $AP < AQ < max(AM, AN)$, but $max(AM, AN)d$, therefore $AQ < d$. This indicates that in spite of adding a point Q to the set, the diameter does not change, which goes against the definition of an inflated set. (Note: although there are other new segments between Q and the other vertices, no segment formed by Q and another vertex will be longer than either that formed with M, or that formed with N).



3. Proposed Theorem for Inflated Sets

3.1. If an inflated set is a subset of another inflated set of the same diameter, then they are equal.

Let us prove this theorem by contradiction.
We assume that $\exists A, B$ two inflated sets so that $d_A = d_B$ and $A \subset B(\,implying\ A \neq B)$. Because $A \subset B$, there exists a point P so that $P \in B$ but $P \notin A$. Let there be $A_1 = A \cup \{P\}$. Since A is inflated, adding any point increases the diameter, so $d_{A_1} = d_A$ (1). Also, $A_1 \subset B$, so $d_{A_1} = d_B$ (2). However, $d_A = d_B$, so the relations (1) and (2) are contradictory. This means that the hypothesis was false. In conclusion, if A and B are two inflated sets so that $d_A = d_B$ and $A \subseteq B$, then it is certain that $A = B$.

4. Conclusion

In conclusion, our article presents both the process of inflating different geometric figures, as well as some rules and ideas related to it. Our explanations on inflating sets offer an insight into the mathematical algorithms that underlie the inflating process.

Notes d'édition

(1) Question : y a-t-il un nombre fini d'étapes ?

(2) A priori, « gonfler une figure » nécessite que la figure de départ soit convexe, ce qui n'est pas le cas d'un disque avec un point extérieur.

(3) Il n'existe pas une façon unique de gonfler une figure : on pourrait imaginer d'autres figures gonflées à partir d'un disque et d'un point extérieur.

(4) Ici encore, on a l'impression que le gonflage du triangle est unique avec la méthode présentée.

(5) La présentation du triangle de Reuleaux arrive un peu tardivement car il a déjà été mentionné plusieurs fois auparavant.

(6) Il n'est peut-être pas si évident qu'ajouter le point F après avoir ajouté le point E n'augmente pas le diamètre de la figure, mais c'est bien le cas.