Generating an octagon

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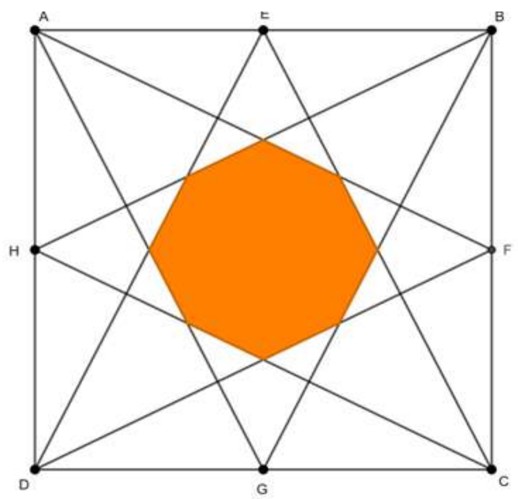
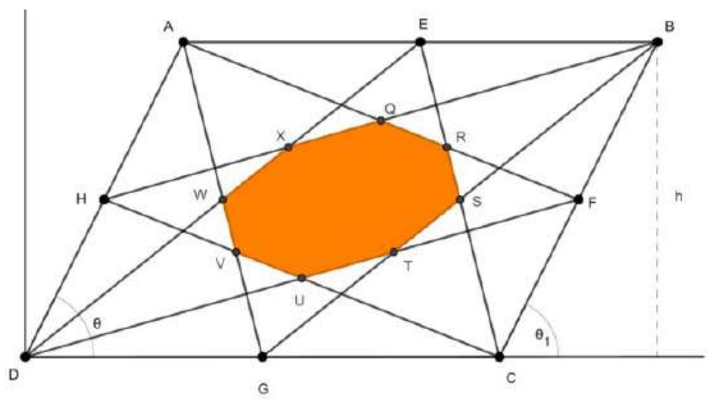
The subject : Let $ABCD$ be a square and E, F, G, H midpoints of its sides. Each midpoint is connected by a line with its opposite side edges.



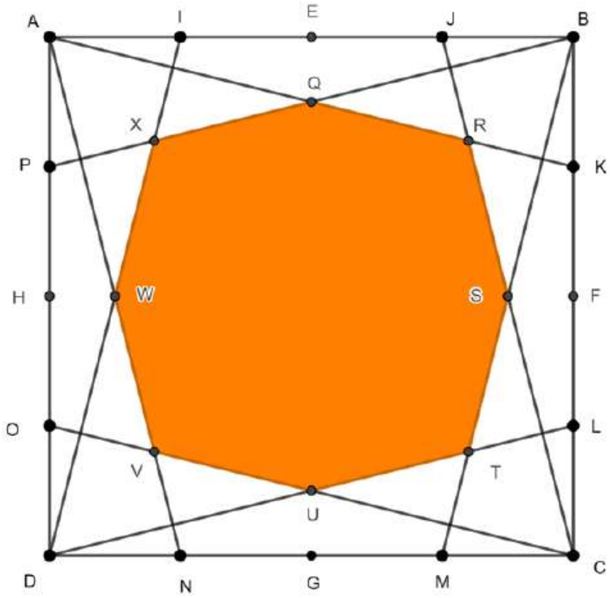
Determine the surface area of the octagon.

Generalizations

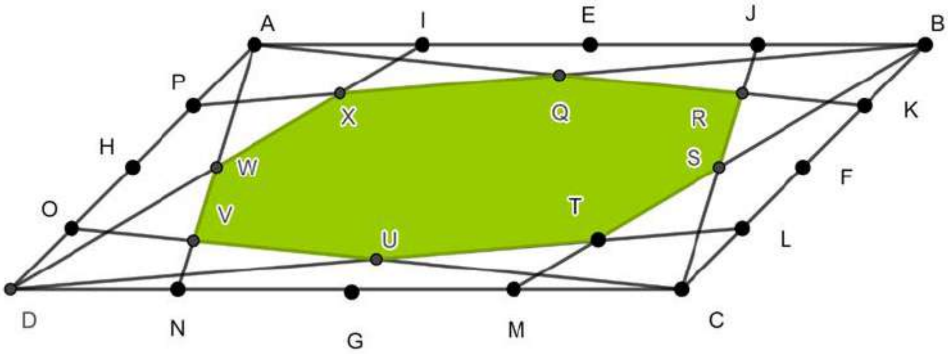
1) Generalize the problem for each parallelogram $ABCD$.



2) But what if the points divide the sides of the square in 3 parts? How about 4? How does the area of the octagon change?



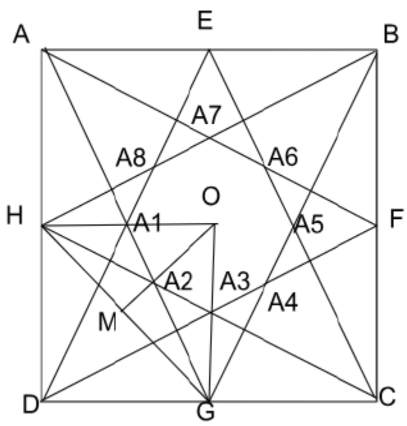
3) How does point 2) generalization apply to a parallelogram?



4) How do you make a regular octagon?

Solution

Let $A_1,\ A_2, A_3, A_4, A_5, A_6, A_7, A_8$ be vertices the of the octagon and $AB = l$, the side of the square.



$AE = \frac{AB}{2} = \frac{l}{2} = \frac{CD}{2} = DG$ | $\Rightarrow AEDG$- rectangle | $\Rightarrow$
$AE \parallel DG$ | $\{A_1\} = DE \cap AG = d_1 \cap d_2$
$\widehat{EAD} = 90°$

$\Rightarrow A_1 \rightarrow$ midpoint of DE and AG. In the same way we demonstrate that A_5 is the midpoint of $EC \Rightarrow$ $A_1A_5 \rightarrow$middle line in $\Delta DEC \Rightarrow A_1A_5 = \frac{l}{2}$. In the same way we demonstrate that $A_3A_7 = \frac{l}{2}$.
$\Rightarrow$
Let $A_1A_5 \cap A_3A_7\{O\}$, O is the center of the octagon
$\Rightarrow A_1O = \frac{l}{4} = A_3O = A_5O = A_7O$ (2)

$A_1 \rightarrow$midpoint of DE, $H \rightarrow$ midpoint of $AD \Rightarrow A_1H$ is the middle line in $\Delta ADE \Rightarrow$
$\Rightarrow A_1H = \frac{AE}{2} = \frac{l}{4} = A_1O \Rightarrow A_1$ is the midpoint of OH.

In the same way we demonstrate that A_2 is the midpoint of $OG \Rightarrow$ In $\triangle GOH$: GA_1 and HA_3 are medians, $GA_1 \cap HA_3 = \{A_2\} \Rightarrow A_2$ is center of gravity in $\triangle GOH$.

Let $OA_2 \cap GH = \{M\} \Rightarrow OA_2 = \frac{2}{3} \cdot OM$, $OM \rightarrow$median
$\triangle GOH$: right isosceles triangle $\Rightarrow \begin{cases} OA_2 \rightarrow \text{bisector } \widehat{GOH} \ (1) \\ OM = \frac{HG}{2} \end{cases} \Rightarrow OA_2 = \frac{HG}{3}$
In $\triangle DGH$: right triangle, $DG = DH = \frac{l}{2} \Rightarrow$ Using the Pythagorean Theorem:
$HG^2 = \frac{l^2}{4} + \frac{l^2}{4} \Rightarrow HG = \frac{\sqrt{2} \cdot l}{2}$. $\Rightarrow$

$\Rightarrow OA_2 = \frac{\sqrt{2} \cdot l}{6}$. In the same way we demonstrate $OA_4 = OA_6 = OA_8 = \frac{\sqrt{2} \cdot l}{6}$ (3).

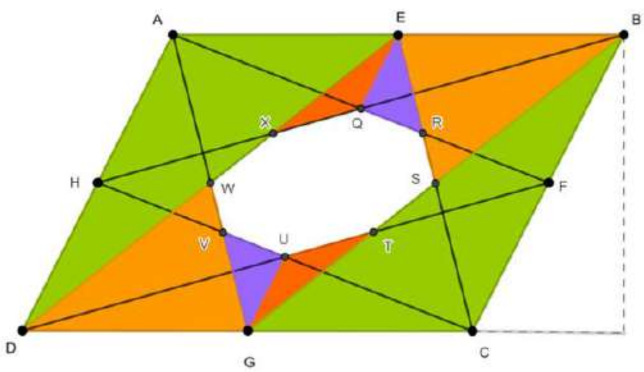
(1) $OA_2 \rightarrow$bisector $\widehat{GOH} = 90^O \Rightarrow \widehat{A_1OA_2} = \widehat{A_2OA_3} = 45^O$. Analog to $\widehat{A_3OA_4} = \widehat{A_4OA_5} = \ldots = \widehat{A_8OA_1} = 45^O$ (4).

From (2) , (3) , (4)$\Rightarrow \triangle A_1OA_2 \equiv \triangle A_2OA_3 \equiv \ldots \equiv \triangle A_8OA_1 \Rightarrow$

$\Rightarrow A_{\Delta A_1OA_2} = A_{\Delta A_2OA_3} = \ldots = A_{\Delta A_8OA_1} = \frac{OA_1 \cdot OA_2 \cdot sin(\widehat{A_1OA_2})}{2} = \frac{\frac{l}{4} \cdot \frac{\sqrt{2} \cdot l}{6} \cdot \frac{\sqrt{2}}{2}}{2} = \frac{l^2}{48} \Rightarrow$

$\Rightarrow A_{octagon} = A_{\Delta} \cdot 8 = 8 \cdot \frac{l^2}{48} = \frac{l^2}{6} = \frac{1}{6} \cdot A_{ABCD}$.

Generalization 1 (1)



$AB||CD \Rightarrow BE||DG$ and $BE = DG \Rightarrow BGDE \rightarrow$ parallelogram $\Rightarrow A_{BEDG} = BE \cdot h = \frac{AB \cdot h}{2} = \frac{A_{ABCD}}{2}$

$BE = \frac{AB}{2} = \frac{CD}{2} = CG$ and $BE||CG \Rightarrow BECG \rightarrow$parallelogram, $CE, BG \rightarrow$ diagonals and $CE \cap BG = \{S\} \Rightarrow S \rightarrow$ midpoint of $BG \Rightarrow SG = \frac{BG}{2}$

Analog to $\begin{cases} W \rightarrow \text{midpoint of } DE \Rightarrow WE = \frac{DE}{2} \\ U, G \rightarrow \text{midpoint of } CH, CD \end{cases}$ $\Rightarrow SG = WE, SG||WE \Rightarrow$

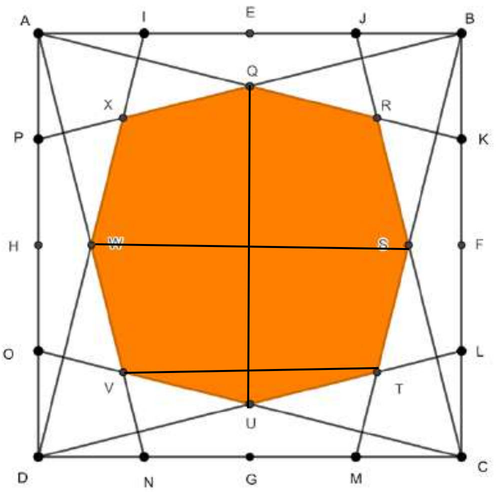
$BG = DE, BG||DE \Rightarrow BEDG \rightarrow$parallelogram

$\Rightarrow SGWE \rightarrow$parallelogram$\Rightarrow A_{SGWE} = SG \cdot d(S, WE) = \frac{BG \cdot d(S, WE)}{2} = \frac{BG \cdot d(B, DE)}{2} =$
$= \frac{A_{BEDG}}{2} = \frac{A_{ABCD}}{4} = A_{octagon} + A_{\triangle VUG} + A_{\triangle GUT} + A_{\triangle EXQ} + A_{\triangle EQR}$ (1)

$U, G \rightarrow$ middle of $CH, CD \Rightarrow UG \rightarrow$ middle line in $\triangle CDH$ and $\Rightarrow UG = \frac{DH}{2} = \frac{AH}{2}$ and $UG||DH \Rightarrow UG||AH$ and $UH \cap AG = \{V\} \Rightarrow$Using the Fundamental Theorem of Similarity: $\triangle AHV \sim \triangle GUV \Rightarrow$
$k = \frac{VG}{AV} = \frac{UV}{HV} = \frac{UG}{AH} = \frac{1}{2} \Rightarrow A_{\triangle GUV} = k^2 \cdot A_{\triangle AVH} = \frac{A_{\triangle AVH}}{4}$
$AH = \frac{AD}{2} = \frac{BC}{2} = CF, AH||CF \Rightarrow AHCF \rightarrow$parallelogram$\Rightarrow A_{AHCF} = AH \cdot d(A, CF) =$
$= \frac{AD \cdot d(A, BC)}{2} = \frac{A_{ABCD}}{2}$.
$A_{\triangle AHC} = \frac{AH \cdot d(C, AH)}{2} = \frac{A_{AHCF}}{2} = \frac{A_{ABCD}}{4}$
In $\triangle ACD$: $AG, CH \rightarrow$medians, $AG \cap CH = \{V\} \Rightarrow V \rightarrow$center of gravity in $\triangle ACD \Rightarrow$
$\Rightarrow HV = \frac{CH}{3} \Rightarrow A_{\triangle AHV} = \frac{HV \cdot d(A, CH)}{2} = \frac{A_{\triangle AHC}}{3} = \frac{A_{ABCD}}{12} \Rightarrow A_{\triangle GUV} = \frac{A_{ABCD}}{48}$.
Analog to $A_{\triangle GUT} = A_{\triangle EXQ} = A_{\triangle EQR} = \frac{A_{ABCD}}{48} = A_{\triangle}$.
Using (1): $A_{octagon} + 4 \cdot A_{\triangle} = \frac{A_{ABCD}}{4} \Rightarrow A_{octagon} = \frac{A_{ABCD}}{4} - \frac{A_{ABCD}}{12} \Rightarrow A_{octagon} = \frac{A_{ABCD}}{6}$.

Generalization 2

Suppose that in the original problem, the segments from the vertices of the square extended not to the midpoints of the opposite sides but to the near-quarter or some other ratio.



E, F, G, H →a quarter from one of the
$\triangle DON$: isosceles right triangle ⇒ $\widehat{DON} = 45^o$
$\triangle ADC$: isosceles right triangle ⇒ $\widehat{DAC} = 45^o$
ON, AC →straight line; OA →secant line ⇒ $\widehat{DON}, \widehat{DAC}$ →corresponding angles ⇒

⇒ $ON || AC$ ⇒ Using the Fundamental Theorem of Similarity: $\triangle DON \sim \triangle DA?$ ⇒ ; $\triangle VON \sim \triangle VCA$ ⇒

$$\Rightarrow \frac{OD}{AD} = \frac{DN}{DC} = \frac{ON}{AC} = \frac{1}{4}$$
$$\Rightarrow \frac{ON}{AC} = \frac{OV}{VC} = \frac{VN}{AV} = \frac{1}{4} \Rightarrow AV = 5 \cdot VN \Rightarrow AN = 5 \cdot VN, VN = \frac{1}{5} \cdot AN$$

$AI = \frac{1}{4} \cdot AB = \frac{1}{4} \cdot l = DN, AI || DN \Rightarrow AIND$ →parallelogram; $DI \cap AN = \{W\}, DI, AN$: diagonals ⇒

⇒ W → the midpoint of $AN \Rightarrow AN = 2 \cdot WN$; $AN = 5 \cdot VN$ ⇒ $WN = 2{,}5 \cdot VN = WV + WN$ ⇒

⇒ $WN = \frac{3}{2} \cdot VN = \frac{3}{10} \cdot AN$
$\triangle ADN$:right triangle⇒Using the Pythagorean Theorem:
$AD^2 + DN^2 = AN^2 = l^2 + \frac{l^2}{16} \Rightarrow AN = \frac{\sqrt{17} \cdot l}{4}$ ⇒ $WV = \frac{3\sqrt{17} \cdot l}{40}$

Analog to $VU = UT = TS = SR = QR = XQ = XW = \frac{3\sqrt{17} \cdot l}{40}$

W, S →midpoints of $AN, BM \Rightarrow WS || CD$; $\frac{VN}{NW} = \frac{TM}{MS} = \frac{2}{3} \Rightarrow VT || WS$ ⇒ $VT || CD$ ⇒

⇒Using the Fundamental Theorem of Similarity:
$\triangle VUT \sim \triangle UCD \Rightarrow K = \frac{VU}{UC} = \frac{UT}{DU} = \frac{VT}{CD} = \frac{3}{5} \Rightarrow \; VT = \frac{3l}{5}$
$A_{\triangle UVT} = K^2 \cdot A_{\triangle UCD} = \frac{9}{25} \cdot \frac{UG \cdot CD}{2}$

W, S →midpoints of $AN, BM \Rightarrow WS = \frac{3l}{4}$. Analog $QU = \frac{3l}{4} \Rightarrow UG = \frac{1}{8} \cdot l \Rightarrow$

$$\Rightarrow A_{\triangle UVT} = \frac{9 \cdot l \cdot \frac{l}{8}}{50} = \frac{9l^2}{400}.$$

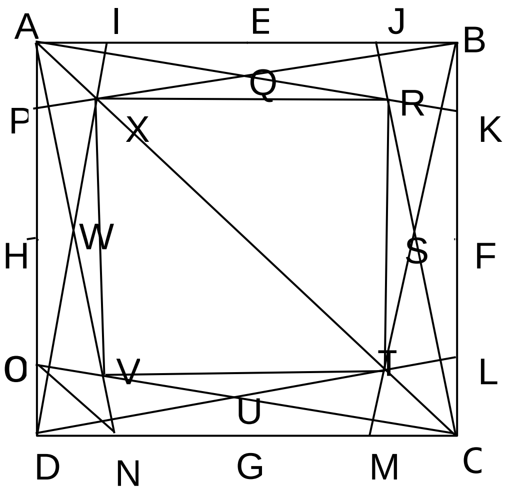
Analog $A_{\triangle RST} = A_{\triangle RQT} = A_{\triangle ?WX} = \frac{9l^2}{400} = A_{\triangle}$.

Analog $RT = RX = XV = \frac{3l}{5}, XR || CD, XV || AD \Rightarrow VTRX$ →square ⇒ $A_{VTRX} = VT^2 = \frac{9l^2}{25}$.

$A_{octagon} = A_{VTRX} + 4 \cdot A_{\triangle} = \frac{9l^2}{25} + \frac{9l^2}{100} = \frac{9}{20} \cdot l^2 = \frac{9}{20} \cdot A_{ABCD}$.

After this latest result, we discovered a rule: if the points are at a distance of $\frac{1}{n} \cdot l$ from the vertices of the square, then the area of the octagon is $A_{octagon} = \frac{(n-1)^2}{n(n+1)} \cdot A_{ABCD}$. For $n = 2$, we obtained $A_{octagon} = \frac{1}{6} \cdot ?_{ABCD}$ and for $n = 4$ we obtained $A_{octagon} = \frac{9}{20} \cdot A_{ABCD}$. So, we tried to demonstrate this rule for any n>2, $n \in \mathbb{R}$.

I, J, K, L, M, N, O, P are at a distance of $\frac{1}{n} \cdot l$ from one of the vertices of the square.



$\triangle DON$: isosceles right triangle $\Rightarrow \widehat{DON} = 45^o$
$\triangle ADC$: isosceles right triangle $\Rightarrow \widehat{DAC} = 45^o$
$ON, AC \rightarrow$straight line
$AD \rightarrow$secant line $\Big\} \Rightarrow \widehat{DON}, \widehat{DAC} \rightarrow$corresponding angles $\Big\} \Rightarrow$

$\Rightarrow ON || AC \Rightarrow$ Using the Fundamental Theorem of Similarity: $\begin{cases} \triangle DON \sim \triangle DAC \Rightarrow \\ \triangle VON \sim \triangle VCA \Rightarrow \end{cases}$

$\Rightarrow K_1 = \frac{OD}{AD} = \frac{DN}{DC} = \frac{ON}{AC} = \frac{\frac{l}{n}}{l} = \frac{1}{n}$

$\Rightarrow K_1' = \frac{ON}{AC} = \frac{OV}{VC} = \frac{VN}{AV} = K = \frac{1}{n} \Rightarrow VA = n \cdot VN \Rightarrow AN = (n+1) \cdot VN, VN = \frac{AN}{n+1}$

$AI = \frac{1}{n} \cdot AB = \frac{1}{n} \cdot CD = DN$
$AB || CD \Rightarrow AI || DN$ $\Big\} \Rightarrow AIND \rightarrow$ parallelogram $AN, DI \rightarrow$ diagonals, $AN \cap DI = \{W\}$ $\Big\} \Rightarrow$

$\Rightarrow W \rightarrow$ midpoint of $AN \Rightarrow WN = \frac{1}{2} \cdot AN = \frac{n+1}{2} \cdot VN \Rightarrow$

$\Rightarrow WV = WN - VN = \frac{n-1}{2} \cdot VN$
$VN = \frac{AN}{n+1}$ $\Big\} \Rightarrow WV = AN \cdot \frac{n-1}{2(n+1)}$

$\triangle ADN$: right triangle $\Rightarrow$ Using the Pythagorean Theorem: $AD^2 + DN^2 = AN^2 = l^2 + \left(\frac{l}{n}\right)^2 = \frac{l^2 \cdot (n^2+1)}{n^2} \Rightarrow AN = \frac{l}{n} \cdot \sqrt{n^2+1} \Rightarrow WV = \frac{l \cdot (n-1) \cdot \sqrt{n^2+1}}{2n \cdot (n+1)}$

Analog $UT = VU = TS = SR = RQ = QX = XW = \frac{l \cdot (n-1) \cdot \sqrt{n^2+1}}{2n \cdot (n+1)}$

$\triangle UVT$:isosceles triangle($VU = UT$)
$\triangle DUC$:isosceles triangle($DU = UC = \frac{DL}{2} = \frac{C\square}{2}$)
$VC \cap DT = \{U\} \Rightarrow \widehat{VUT} \equiv \widehat{DUC}$ $\Big\} \Rightarrow \triangle UVT \sim \triangle UDC \Rightarrow$

$\Rightarrow K_2 = \frac{UV}{UD} = \frac{UT}{UC} = \frac{VT}{CD} = \frac{\frac{n-1}{2 \cdot (n+1)} \cdot CO}{\frac{1}{2} \cdot CO} = \frac{n-1}{n+1} \Rightarrow A_{VUT} = A_{DUC} \cdot K_2^2 = \frac{(n-1)^2}{(n+1)^2} \cdot A_{DUC}, VT = \frac{l \cdot (n-1)}{n+1}$
$U, G \rightarrow midpoints\ of\ CD, CO \Rightarrow UG = \frac{DO}{2} = \frac{l}{2n} \Rightarrow A_{DUC} = \frac{CD \cdot UG}{2} = \frac{l^2}{4n}$ $\Big\} \Rightarrow$

$\Rightarrow A_{VUT} = \frac{l^2 \cdot (n-1)^2}{4n \cdot (n+1)^2}$

Analog for $A_{RST} = A_{RQX} = A_{XWV} = \frac{l^2\cdot(n-1)^2}{4n\cdot(□+1)^2}$.

Analog $TR = RX = XV = \frac{l(n-1)}{n+1} \Rightarrow VTRX \rightarrow$ rhombus $\Big\} \Rightarrow$

$\triangle\, UVT \sim \triangle\, UDC \Rightarrow \widehat{UVT} \equiv \widehat{UCD}$ (internal alternate angles) $\Big\} \Rightarrow VT || CD \Rightarrow VT || TR$

Analog $TR || BC, BC \perp CD$

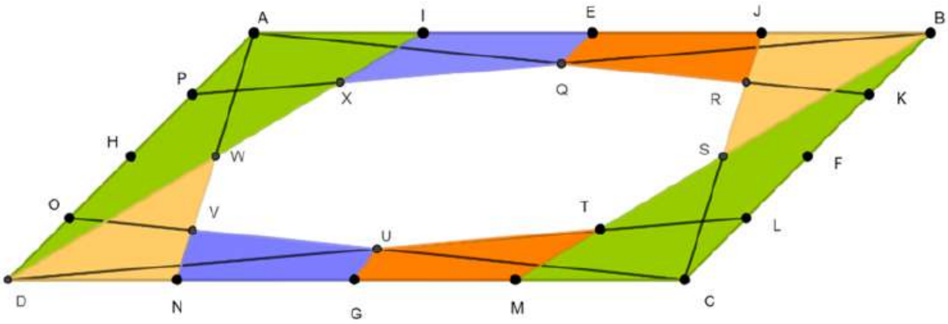
$\Rightarrow VTRX \rightarrow$ square $\Rightarrow A_{VTRX} = (VT)^2 = \frac{l^2\cdot(n-1)^2}{(n+1)^2}$

$A_{octagon} = 4\cdot A_{VUT} + A_{VTRX} = \frac{l^2\cdot(n-1)^2}{n\cdot(n+1)^2} + \frac{l^2\cdot(n-1)^2}{(n+1)^2} = \frac{l^2\cdot(n-1)^2}{n\cdot(n+1)} = \frac{(n-1)^2}{n\cdot(n+1)}\cdot A_{ABCD}$

$(A_{ABCD} = l^2)$.

Generalization 3

Since the original problem on the square turned out to be true for any parallelogram, the natural question at this point is to ask whether this latest result generalizes to any parallelogram.



$AI = □J = CM = DN = \frac{1}{n}\cdot AB = \frac{1}{n}\cdot CD$

$AP = DO = CL = BK = \frac{1}{n}\cdot AD = \frac{1}{n}\cdot BC$

$\frac{DN}{CD} = \frac{DO}{AD} = \frac{1}{n}$ ⇒Using the Reciprocal of Thales' Theorem: $ON || AC$ ⇒Using the Fundamental Theorem of Similarity: $\left\{ \begin{array}{l} \triangle\, DON \sim \triangle\, DAC \\ \triangle\, VON \sim \triangle\, VCA \end{array} \right\} \Rightarrow$

$\Rightarrow k_1 = \frac{DN}{CD} = \frac{DO}{AD} = \frac{ON}{AC} = \frac{1}{n}$ and $k_2 = \frac{VO}{VC} = \frac{VN}{VA} = \frac{ON}{AC} = \frac{1}{n} \Rightarrow VA = \frac{1}{n}\cdot VN \Rightarrow$

$\Rightarrow AN = (n+1)\cdot VN, VN = \frac{AN}{n+1}$.

$AI = \frac{1}{n}\cdot AB = \frac{1}{n}\cdot □D = DN, AI || DN \Rightarrow AIDN \rightarrow$parallelogram $\Big\} \Rightarrow$

$AN, DI \rightarrow$diagonals and $AN \cap DI = \{W\}$

$\Rightarrow W \rightarrow$midpoint of $AN, DI \Rightarrow WN = \frac{AN}{2} = \frac{n+1}{2}\cdot VN \Rightarrow WV = \frac{n-1}{2}\cdot VN = \frac{n-1}{n+1}\cdot AW$

In the same way, we demonstrate $WX = \frac{n-1}{n+1}\cdot DW$.

$\frac{WX}{DW} = \frac{W□}{AW} = \frac{n-1}{n+1}$ $\Big\} \Rightarrow \triangle\, ADW \sim \triangle\, VWX \Rightarrow$

$\widehat{XWV} = \widehat{AWD}$ (opposite angles at the apex)

$$\Rightarrow k_3 = \frac{WV}{AW} = \frac{WX}{DW} = \frac{VX}{AD} = \frac{n-1}{n+1} \Rightarrow \begin{cases} A_{\triangle VWX} = (k_3)^2 \cdot A_{\triangle ADW} = (\frac{n-1}{n+1})^2 \cdot A_{\triangle ADW} \\ VX = \frac{n-1}{n+1} \cdot AD. \text{ Analog to } RT = \frac{n-1}{n+1} \cdot BC \text{ and} \end{cases}$$

$VT = RX = \frac{n-1}{n+1} \cdot AB.$

$AN \cap DI = \{W\} \Rightarrow d(W, AD) = \frac{1}{2} \cdot d(N, AD) = \frac{1}{2} \cdot \frac{1}{n} \cdot d(C, AD) \Rightarrow$

$\Rightarrow A_{\triangle ADW} = \frac{AD \cdot d(W,AD)}{2} = \frac{1}{4n} \cdot AD \cdot d(C, AD) = \frac{1}{4n} \cdot A_{ABCD} \Rightarrow A_{\triangle VWX} = \frac{(n-1)^2}{4n(n+1)^2} \cdot A_{ABCD}$

Analog to $A_{\triangle XQR} = A_{\triangle RST} = A_{\triangle VUT} = \frac{(n-1)^2}{4n(n+1)^2} \cdot A_{ABCD} = A_{\triangle}.$

$\triangle ADW \sim \triangle VWX \Rightarrow \widehat{XVW} = \widehat{WAD}$ (alternate internal angles, AV →secant)⇒

$\Rightarrow VX || AD$. In the same way, we demonstrate $VT || CD \Rightarrow \widehat{ADC} = \widehat{XVT} \Rightarrow$

$\Rightarrow sin(\widehat{ADC}) = sin(\widehat{XVT}).$

$VT = XR$ and $VX = RT \Rightarrow VTRX$ →parallelogram⇒ $A_{VTRX} = VT \cdot VX \cdot sin(\widehat{TVX}) \Rightarrow$

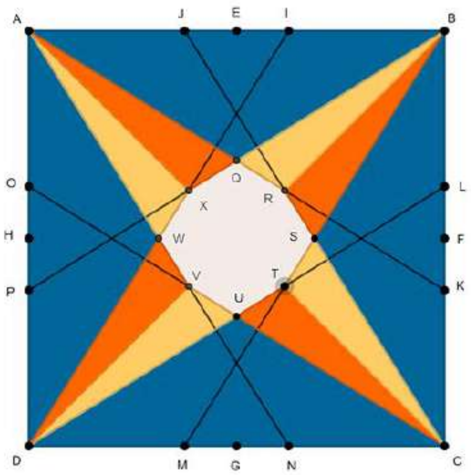
$\Rightarrow A_{VTRX} = \frac{(n-1)^2}{(n+1)^2} \cdot A_{ABCD}.$

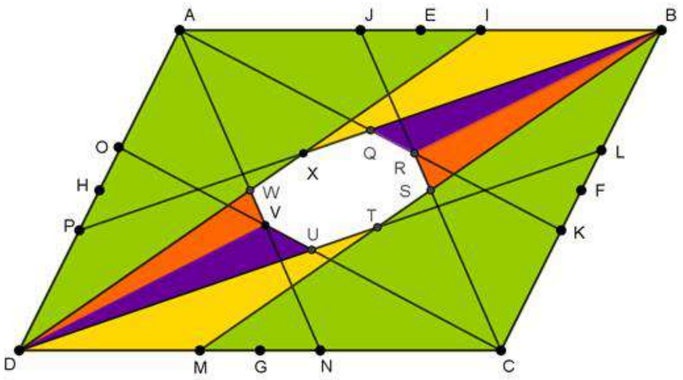
$A_{octagon} = 4 \cdot A_{\triangle} + A_{VTRX} = 4 \cdot \frac{(n-1)^2}{4n(n+1)^2} \cdot A_{ABCD} + \frac{(n-1)^2}{(n+1)^2} \cdot A_{ABCD} \Rightarrow$

$\Rightarrow A_{octagon} = \frac{(n-1)^2}{n(n+1)} \cdot A_{ABCD}.$

Observation: $n - 1$ must be greater than 0 because $VT = \frac{n-1}{n+1} \cdot AB$. As a result, n must be greater than 1. But what happens if $n \in (1,2)$?

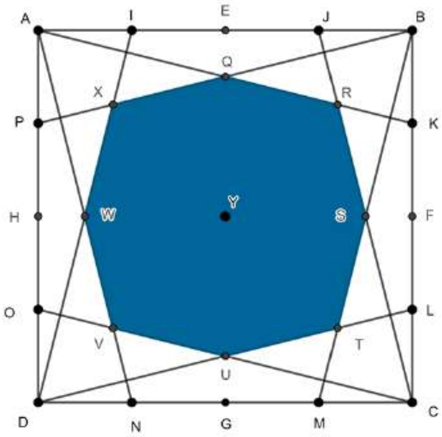
As n decreases between 2 and 1, we find that the pairs of segments like DI and CJ cross and that the area of the octagon continues to shrink as n approaches 1. But surprisingly, for both the square and the parallelogram, none of the ratios and areas change from the solution to the problem. The "overlapping" does not affect the steps in solving the problem.





Generalization 4

When we first thought about how to solve this problem, we incorrectly believed that the initial octagon was a regular octagon, when, in fact, it is not. Although the octagon is equilateral, one can verify that the distances of points Q, S, U, W from the center of the octagon are not equal to the distances of points R, T, V, X from the center. So, we may reasonably ask under what conditions the octagon formed is regular.



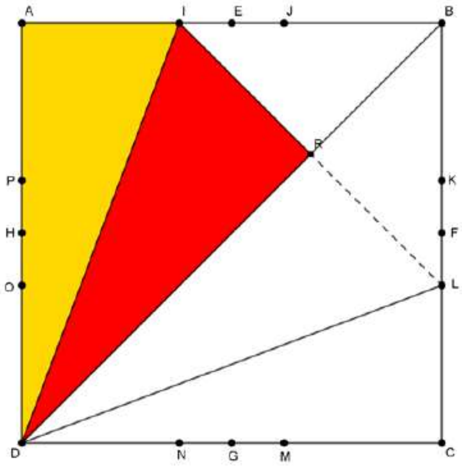
The octagon can be regular only in the case of the square but not for the general parallelogram. From the symmetries of the square, we can establish without difficulty that the octagon is equilateral and that the eight central angles with vertices at Y are all 45^O; however, in general, $QY = SY = UY = WY$ and $RY = TY = VY = XY$, but the two sets of segments are not equal to each other. For the octagon to be regular, all vertex-center distances must be equal, so we consider the case of $WY = VY$.

In $\triangle ADN$: $H, W \rightarrow$ midpoints of $AD, AN \Rightarrow HW \rightarrow$ middle line $\Rightarrow HW = \frac{DN}{2} = \frac{l}{2n}$ and $HY = \frac{CD}{2} = \frac{l}{2} \Rightarrow WY = \frac{n-1}{2n} \cdot l$ (1)

Using these results: $VT || CD$ and $VT = \frac{n-1}{n+1} \cdot CD$ that were already found in the previous demonstrations, we have $\triangle YVT \sim \triangle YDC$, where $k = \frac{n-1}{n+1} \Rightarrow$

$VY = \frac{n-1}{n+1} \cdot DY = \frac{n-1}{2(n+1)} \cdot BD = \frac{\sqrt{2}(n-1)}{2(n+1)} \cdot l$ (2)

From (1), (2) and $WY = VY \Rightarrow \frac{n-1}{2n} = \frac{\sqrt{2}(n-1)}{2(n+1)} \Rightarrow n = \frac{1}{\sqrt{2}-1}$.



The desired points for which $n = \frac{1}{\sqrt{2}-1}$ are found by bisecting the 45 degree angles between the sides of the square and the diagonals.These lines can also be found by reflecting each of the triangles equivalent to $\triangle DAI$ onto the diagonal, as illustrated.

Editing notes

(1) Generalization 1 can be obtained without computation from the case of the square. In fact, the parallelogram can be turned into a square by applying a dilatation (which multiplies all areas by the same factor) and a transvection (which preserves all areas). This also applies to Generalization 3 which follows by the same argument from Generalization 2.