Game of Life on Various Tilings

Year 2023– 2024

Alexandru Tudor Ioniță, Ilinca Istrate, Yannis Ștefănuț Lițcăneanu Mînzat, Radu Mihail Nistor, Alex Florin Oprea, Adrian Solcanu, Victor Solcanu, students in 9th grade

School: Costache Negruzzi National College, Iaşi

Teachers: Adrian Zanoschi, Costache Negruzzi National College, Iaşi, and Vlad Tuchiluș, DPhil student, University of Oxford

Researcher: Vlad Tuchiluș, DPhil student, University of Oxford

Abstract

Inspired by the Game of Life, created by John Horton Conway, we try to find visually and mathematically interesting patterns formed by using different types of tiles and sets of rules. Not only we deeply dive into understanding the figures that appear when the tiles have a hexagonal or a triangular shape better, but we also focus on the 1D case, which gives rise to fractal patterns and the Sierpinski triangle. Using the opportunity of creating our own rules of the Game and utilizing a computer program, we managed to find out how interesting patterns behave on different types of tiles: some patterns are static, some are oscillatory and some of them even glide.

1. INTRODUCTION

The mathematician John Horton Conway proposed the following game in 1970, called the Game of Life: Consider an infinite grid of squares, which we call cells. Each cell can be in one of two states: alive or dead. Every cell has 8 neighbours, which are the squares with which it shares at least a side or a vertex. Every second, the following happen to all cells:

- Any living cell with fewer than two living neighbours dies, as if by underpopulation.
- Any living cell with two or three living neighbours lives on to the next generation.
- Any living cell with more than three living neighbours dies, as if by overpopulation.
- Any dead cell with exactly three living neighbours becomes a living cell, as if by reproduction.

Depending on the initial distribution of living cells, different interesting patterns emerge:

• Some patterns are stable, meaning they do not change at all e.g. a 2×2 block of cells.

• Some patterns oscillate, meaning that after a certain period of time they return to the original pattern and repeat e.g. a 1×3 block of cells, which repeats with period 2.

• Some patterns called spaceships glide across the plane, meaning that after a certain period of time the same pattern reappears but at a different position in the plane.

The choice of a square grid of cells is arbitrary. Different tilings of the plane could be used such as a triangular tiling, hexagonal tiling, etc. The students' research question would be to devise new rules for a Game of Life on different types of tilings and explore what patterns emerge.

We begin our search for answers in section 2, in which we explain in a more detailed manner what rules, cells and tilings represent.

Then, in section 3, we start to deeply analyse each type of tiling we find interesting, putting emphasis on captivating mathematical methods as well as giving examples of some beautiful patterns that we were able to create. In section 3.1 we tackle the subject of a hexagonal grid, using and modifying the computer program that we found, which led us to exploring the vast possibilities that appear in this context.

After that, in section 3.2, we focus on better understanding and discovering interesting patterns that occur when the tiles have a triangular shape. Even though, as cells, triangles have more neighbours than squares or hexagons, it was a pleasure for us to examine and discover rules that lead to unique patterns.

In the last part of our presentation, section 3.3, we dive into the case in which Game of Life functions in just one dimension (1D), following with a thorough explanation of the Sierpiński Triangle.

2. TILINGS AND RULES

The Game of Life consists of a grid made out of cells that are in one of two states, either alive or dead. The way it is determined if a cell becomes alive or dead depends on the initial configuration that the user himself sets and the rule he chooses.

A rule is an algorithm that determines the change in the state of the cell (spawns, survives, remains dead, dies) based on the number of living neighbours that the cell has.

In table 1, we shall explain the mechanism behind the basic rule for squares: a living cell survives if it has either 2 or 3 living neighbours, while a dead cell is born only if it has exactly 3 living neighbours, otherwise nothing changes.

The geometrical shape of each cell can vary, influencing its number of neighbours. Therefore, the Game of Life behaves differently on different tilings of the plane. The only restrictions are that there must be no gaps on the 'board'. Knowing these concepts, we can easily distinguish 2 types of tilings: regular and irregular ones.

For a living cell		For a dead cell	
# living neighbours	Resulting state	# living neighbours	Resulting state
0	Dead	0	Dead
1	Dead	1	Dead
2	Alive	2	Dead
3	Alive	3	Alive
4	Dead	4	Dead
5	Dead	5	Dead
6	Dead	6	Dead
7	Dead	7	Dead
8	Dead	8	Dead

Table 1 Cell state variation in the original Game of Life

A regular tiling is only made out of the same regular shape having the same size across the board, while an irregular tiling contains 2 or more geometrical shapes or the same shape with different sizes (1). The next example in figure 1 is of a famous irregular tiling, the Penrose tiling.

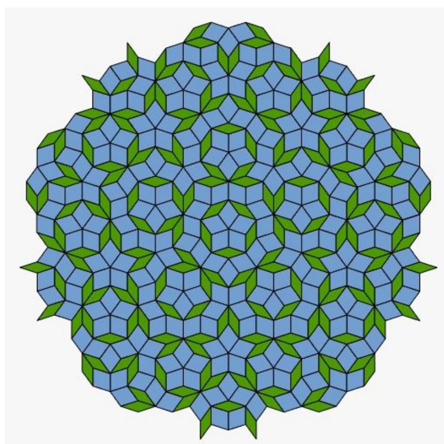


Figure 1 The Penrose Tiling[1]

One of our first tasks was to get familiar with the different geometrical shapes that a regular tiling can have. As we shall prove below, there are only 3 possible regular tilings of the plane.

The first one is the square, which is the most popular and researched one, thus also being the one which gave us the least things to discover. As you can see, the square has 8 neighbours and its arrangement on the 'board' is the simplest out of the other shapes.

[1] The Penrose Tiling, https://en.wikipedia.org/wiki/Penrose_tiling

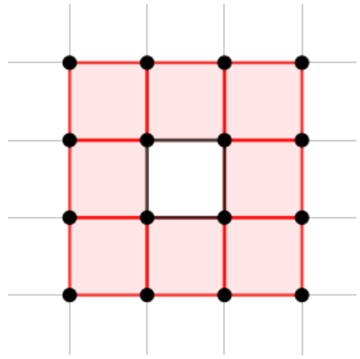


Figure 2 A square and its neighbours

The second one is the hexagon. It is the one with the least neighbours, 6, and also not nearly as researched as the square, which makes it a great candidate for our experiments.

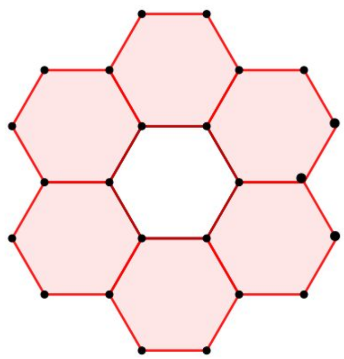


Figure 3 A hexagon and its neighbours

The third one is the triangle, which has the most neighbours, 12, so it's the most complex, leading to very chaotic configurations and motion.

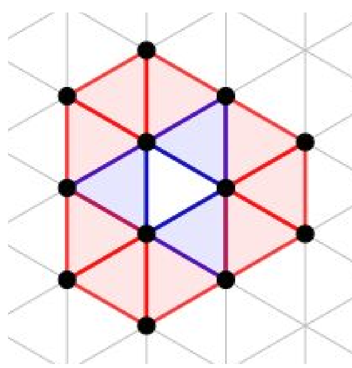


Figure 4 A triangle and its neighbours

We now prove that these are all of the possible regular polygons that can fill our 'board' without gaps.

Proof. Each angle of a regular n-gon has $\frac{(n-2)\cdot 180}{n}$ degrees, by the well-known formula. Let's assume that at each vertex, k n-gons meet (this number must be constant since the n-gons are regular). Then, the value of k is

$$\frac{360}{\frac{(n-2)\cdot 180}{n}} = \frac{2n}{n-2},$$

which must be an integer. We get that $n-2$ divides $2n$. Since we know it divides $2n$ - 4, we get that it is a divisor of 4, which can be either 1, 2, or 4. Therefore, n can only have one of the values 3, 4 or 6.

3. DISCUSSION OF DIFFERENT RULES ON DIFFERENT TILINGS

3.1. Game of Life on hexagons

We begin our discussion of the behaviour of various rules on different tilings with the hexagonal tiling.

Given the fact that working with hexagons on paper is not the easiest of tasks, we have immediately tried to find a solution to switch our work to a computer, where we can save time and effort. After a bit of research, we have managed to find the site https://arunarjunakani.github.io/HexagonalGameOfLife/.

We really have to give a lot of credit to this site as it made our work much more precise and faster. With just a simple click of a button, we could analyse an entire case.

Then, we realized that the site only offered us one rule to work with, which was obviously too little for us. So, our next dilemma was how to change the rules from the site, as it was clear that trying to make a similar site would be quite impossible with our coding knowledge.

Even though it took quite a while to work out how to do it, once we mastered that, it became a piece of cake to change to other rules. Our procedure was the following:

After downloading the code, we would open the file that allows us to change the rules. This file is separated from the animations, so it is really accessible to alter it (2). The complexity of the code helps us switch to different rules within seconds, as we need to change just some numbers and not the whole structure. For example, we present in figure 5 the change in the code from the rule on the site to the rule Survive 3, 4 - Spawn 2, 3. We have used here concise notation to describe rules, which is explained below.

```
if(this.isAlive) {
      if(living < 2 || living > 2) {
            this.shouldSwap = true;
      }
else {
      if (living == 2) {
            this.shouldSwap = true;
      }
```

↓

```
if(this.isAlive) {
      if(living < 3 || living > 4) {
            this.shouldSwap = true;
      }
else {
      if (living == 3 || living == 2) {
            this.shouldSwap = true;
      }
```

Figure 5 Change in the code for hexagons

An interesting initial question was to count the total number of possible rules on the hexagonal tiling, to see whether we had any chance to analyse them all. In order to do this, we have used the fact there exists a bijection between each possible rule for how many neighbours a cell needs to survive and each subset of the set $\{0, 1, 2, \ldots, 6\}$. If a number is in the subset, then a cell with that exact number of living neighbours survives. For any other number of neighbours, it dies. Similarly goes for the number of rules that dictate whether a dead cell comes to life. Therefore, for each part, there are 2^7 possible choices. In total, the hexagonal Game of Life can have one of $2^{14} = 16384$ rules – too many to analyse them all.

We can now discuss complementary rules. For a subset A of a set M, let the complementary of A be A' be the subset $M \setminus A$. For a rule R1 *Spawn* A, *Survive* B, let the rule *Spawn* B', *Survive* A' be called the complement of R1. These rules have the property that they generate identical behaviours if they are applied to complementary colourings of the tiling (each dead cell becomes alive, and each living cell becomes dead). The only rules that do not have a complement are the ones for which $A' = B$, as they become equivalent. For each subset A there is exactly one rule identical with its complement with *Spawn* A. Therefore, the number of rules identical to their complement is equal to the number of subsets of $\{0, 1, 2, \ldots, 6\}$ and is equal to 2^7. Therefore, there are only $\frac{2^{14}-2^7}{2} + 2^7 = 2^{13} - 2^6 + 2^7 = 8256$ if we consider complementary rules to be in fact equal.

Furthermore, we consider that, by Conway's original motivation for the game, a rule only makes sense if both the subset that determines the *Spawn* and *Survive* rules are continuous i.e. if $a < b < c$ and a and c are in the subset then b must be in the subset. The number of continuous 1-element subsets of $\{0, 1, 2, \ldots, 6\}$ is 7. Similarly, there are 6 2-element continuous subsets, 5 3-element subsets and, so on, 1 7-element subset. The total is $1+\cdots+2+\cdots+7=\frac{7\cdot 8}{2}=28$. In total, there are $28^2 = 784$ continuous rules, which is a significant reduction, but still too many to analyse.

Now we are going to move on to the proper analysis of two specific rules, which we have picked since they exhibit different behaviours.

Our first idea for characterising the behaviour of different rules was to make a list of the simple shapes, made up by either 1, 2, 3, or 4 connected hexagons and analyse their behaviour. If a majority of these shapes had similar behaviour, then we could infer that the rule had a tendency to generate that type of behaviour e.g. periodic or stable shapes.

These simple shapes are called monominoes, dominoes, triominoes and tetrominoes. Together, we will refer to them as polyominoes. We have listed them below:

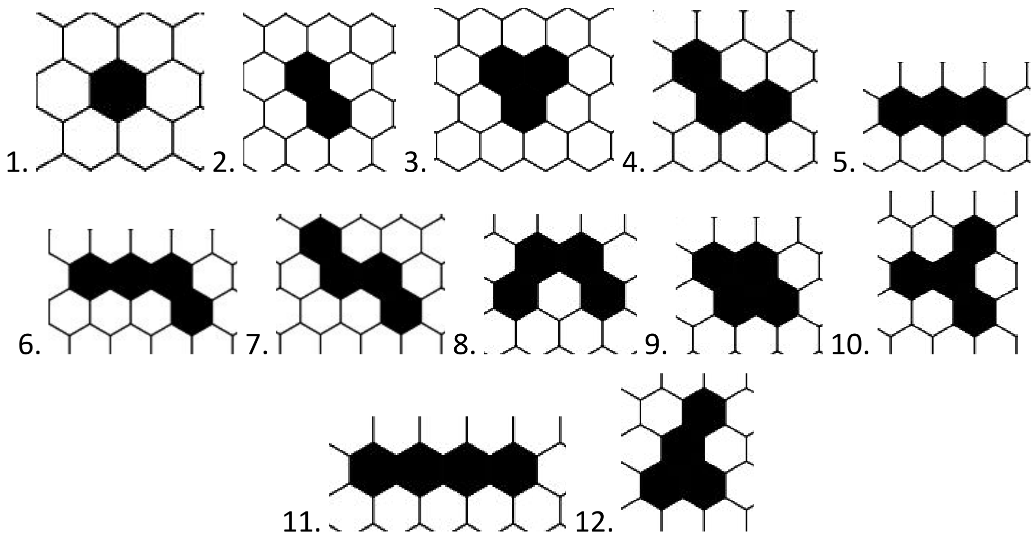


Figure 6 The simple hexagonal shapes

The shapes above are all such simple shapes up to reflections and rotations. During our work we have used the fact that neither reflections nor rotations influence the behaviour of the shapes because the rules only depend on the immediate neighbours (3), not the orientation of the pattern. Symmetric configurations can lead to aesthetically pleasing or harmonious patterns, but the fundamental dynamics are still determined by the local interactions of cells in the grid.

Another way of characterising the behaviour of a rule is by generating random initial shapes, letting the game run, and observing what patterns emerge. On the site there is a button that gives all cells one of the two states with a certain probability. Playing the Game of Life, with time, shapes initially formed either disappear, stabilize or become periodical, thus showing the trend of the rule. For example, if a rule has many periodical shapes, we shall call it a periodical rule.

The first rule that we analysed states that:

- For a cell to be spawned, it needs to have exactly three living neighbours;
- For a living cell to stay alive, it needs to have two or three living neighbours.

It is important to be stated that most of the shapes with interesting properties that we found for this rule are stable and only a small part of them are periodic. We have found around 40 different shapes with interesting properties and at least 30 of them are stable. Thus, this first rule can be considered a stable rule. Using the randomisation technique explained above also confirms this behaviour.

This is somewhat reflected by the polyominoes above. The only ones with interesting properties are numbers 3 and 9 from figure 6, which are stable, and 10, which is 2-step periodic (4).

Figure 7 presents some of the most interesting configurations we discovered. It particularly highlights how one can easily find large stable configurations for this rule.

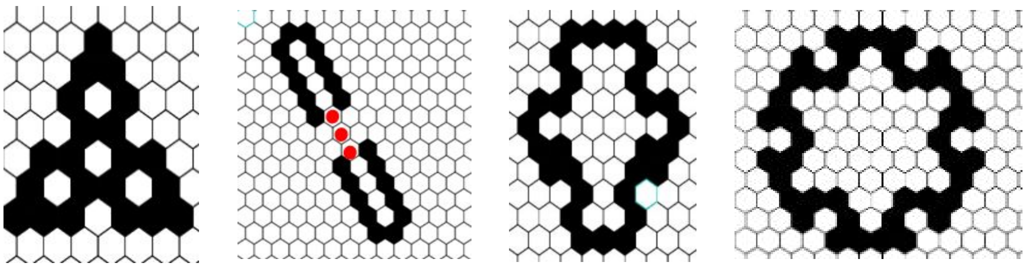


Figure 7 Stable shapes in hexagonal rule 1

The second rule that we analysed states that:

- For a cell to be spawned, it needs to have two or three living neighbours;
- For a living cell to stay alive, it needs to have three or four living neighbours.

Contrary to the first rule, most of the shapes that we found are periodic and only a few of them are stable. This is again confirmed by the randomisation technique. Thus, this rule can be considered a periodic.

This is also reflected amongst the polyominoes, as number 8 from figure 6 is 4-step periodic, while numbers 2, 3 and 10 are all 2-step periodic (5).

Below we highlight some interesting periodic configurations. The highest period that we have found for a shape is 16 for the following initial configuration:

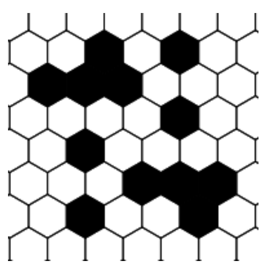


Figure 8 The initial configuration of a 16-step periodic shape

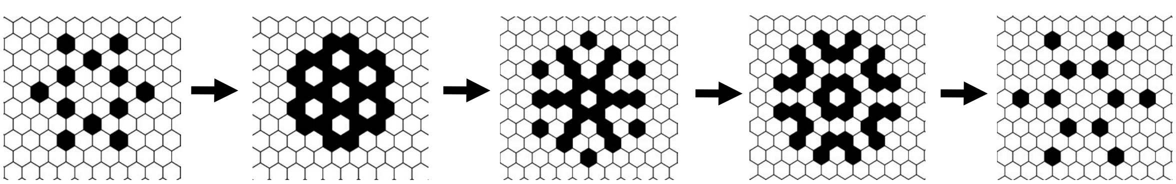


Figure 9 A 5-step periodic shape in hexagonal rule 2

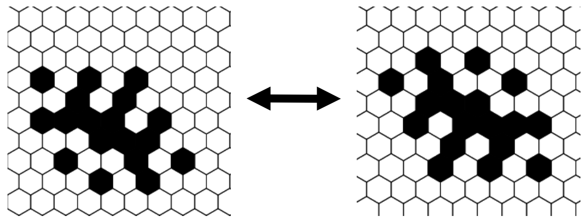


Figure 10 A 2-step periodic shape in hexagonal rule 2

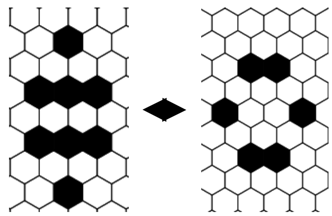


Figure 11 A 2-step periodic shape in hexagonal rule 2

Thus, these two rules which we have discussed are a great example of how different rules can make it harder or easier to find shapes with certain behaviour.

3.2. Game of Life on triangles

The original Game of Life is played on a tiling of the plane using squares and has been studied by many. After having analysed the Game of Life on a hexagonal tile of the plane, we shall now move on to the only regular tiling left, as we have already proved, the one using equilateral triangles.

On such a tiling, each cell has 3 neighbours that share a side (the blue ones in figure 12) and 9 that share a vertex (the red ones in figure 12). Because in Conway's original game, a neighbour was a cell sharing either a side or a vertex, we shall consider each triangle has exactly 12 neighbours. Having so many neighbours, especially compared to the hexagons, this suggests that the Game of Life is much more vivid on a triangular tiling if an appropriate set of rules is chosen. But how many possible rules are there? This is the following part of our analysis.

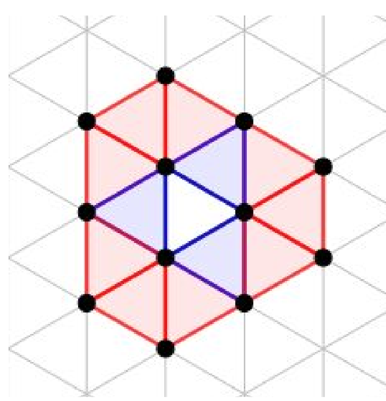


Figure 12 The neighbours in a triangular tiling

In what regards the number of possible rules, we will use a method analogous to the one used for hexagons. We do similar calculations, but the set used is $\{0, 1, 2, \dots, 12\}$. We easily get that there are 2^{26} rules, out of which 2^{13} are identical to their complement. The final result is

$$\frac{2^{26}-2^{13}}{2}+2^{13}=2^{25}-2^{12}+2^{13}=33{,}558{,}528.$$

For counting continuous rules, we proceed just like we did for hexagons. The number of continuous subsets is $\frac{13\cdot 14}{2}=91$, and the resulting total of continuous rules is $91^2=8281$.

We started out our analysis, just like we did for the hexagonal Game of Life, discovering the basic shapes, namely the connected monomino, dominoes and triominoes, which we shall present in the following figure.

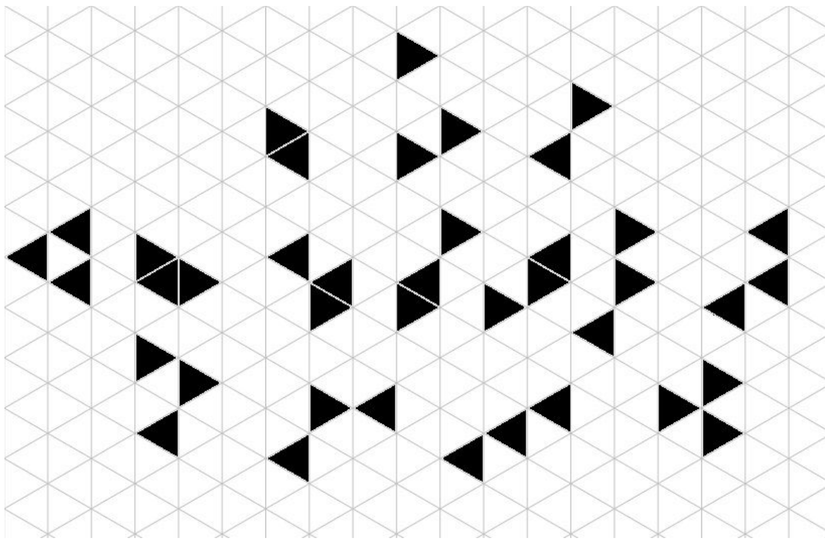


Figure 13 Simple shapes on a triangular tiling

From here, we can see that in a triangular tiling of the plane, there is one possible monomino, 3 possible dominoes, and 11 possible triominoes. We have ignored the shapes that can be obtained from the ones in the figure by either reflections, rotations or both, because they will obviously generate similar behaviours, only with a different orientation.

We have quickly realized that analysing multiple rules manually, on paper, would take a very long time and would imply a great risk of making mistakes. So, it was not feasible. Therefore, we started looking for a code similar to the one used for hexagons. We have come across a YouTube video[2] presenting some gliders in “Triangular Conway’s Game of Life”, as the video was called. Checking the comments, we discovered someone with our exact question: is there a code for it we can use? Gladly, the answer was yes.

The writer of the code, Chase Marangu, had made an editable version[3] on openprocessing.org. The code made our work much easier and we have to give credit to the writer. With the aid of the code, we could now analyse different shapes very quickly. We used the same rule as the writer himself: a living cell only survives if it has either 4, 5 or 6 neighbours, while a dead cell is born when it has either 4 or 8 living neighbours (6).

[2] https://www.youtube.com/watch?v=VOQrDh6AvYQ

[3] https://openprocessing.org/sketch/806868

After checking the 15 shapes in Figure 13, we discovered they all "die" after just a few steps (no living cells are left). Of course, this is obvious since they all are made up of fewer than 4 cells so no cell can have 4 living neighbours to remain or become alive.

In our search of shapes that are either stable, periodic or gliding, we have used a very important characteristic of all of them: if you let the game run on them, they never disappear and can be easily observed. Therefore, we tried using the "random" function of the code. After randomizing a colouring of the plane (each cell is coloured with a certain fixed probability), we let the game run. After it sorts out, we are left with shapes of the three types we are looking for. Although this method doesn't give many results and lots of them repeat quite often, it works and it helped us find the following interesting shapes, including a gliding shape.

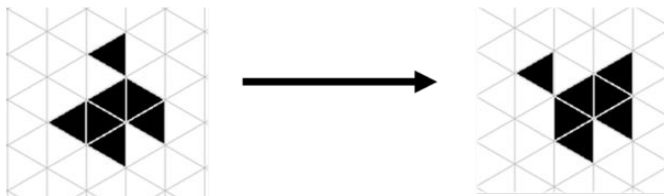


Figure 14 2-step periodic shape

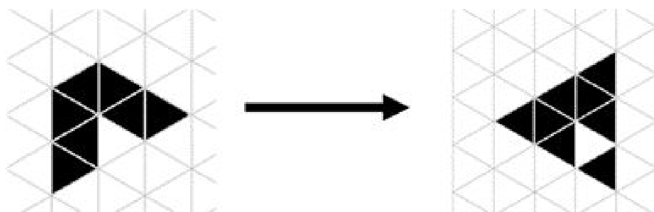


Figure 15 2-step periodic shape

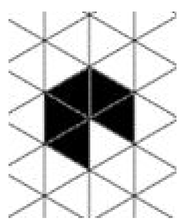


Figure 16 A stable shape

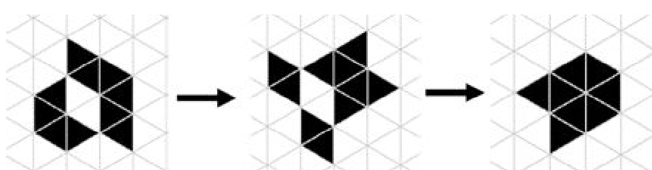


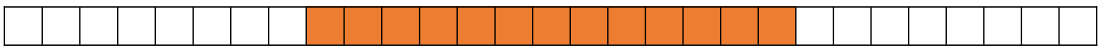
Figure 17 3-step glider

3.3. Game of Life in one dimension

Up until this point of our presentation, we have analysed the Game of Life in two dimensions. In 2D it was very difficult to analyse the cell's evolution exactly. Now, if we take a step higher and play the Game of Life in three dimensions, we will probably lose any control we have left. However, if we go down to a single dimension, we can easily go through multiple stages of the game. In some cases, we can even find the actual pattern.

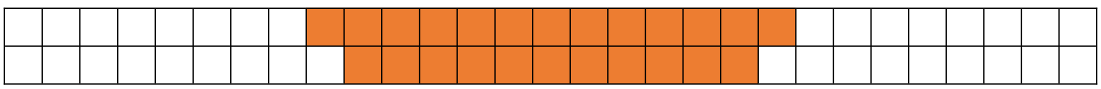
The first major advantage of being in a single dimension is that every cell can have no more than two neighbours. That is because all of our tilings are represented on a straight line. This obviously limits the number of rules we can have.

We did not know exactly where to start, but in this situation, we had the possibility to briefly analyse almost every single rule so that we find something interesting. At first, we chose the rule where a cell is born if it has two living neighbours and stays alive if it has, again, two living neighbours. We have managed to analyse and understand a significant part of its behaviour, which can be observed in the following examples. But first, let us explain to you what an example actually represents.

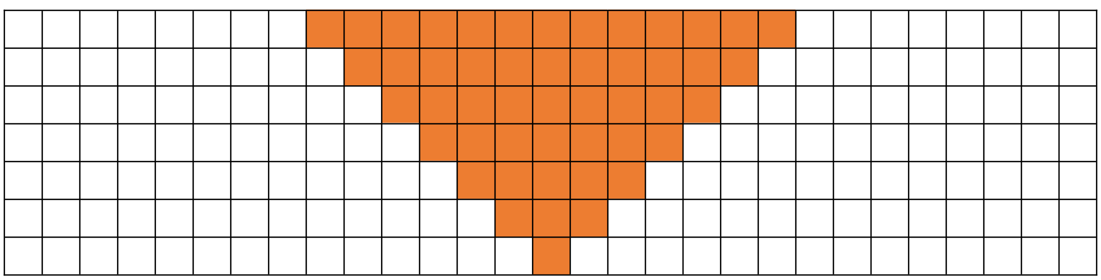


This is the first row of our example. Here we have our starting configuration of cells on the line that are going to evolve.

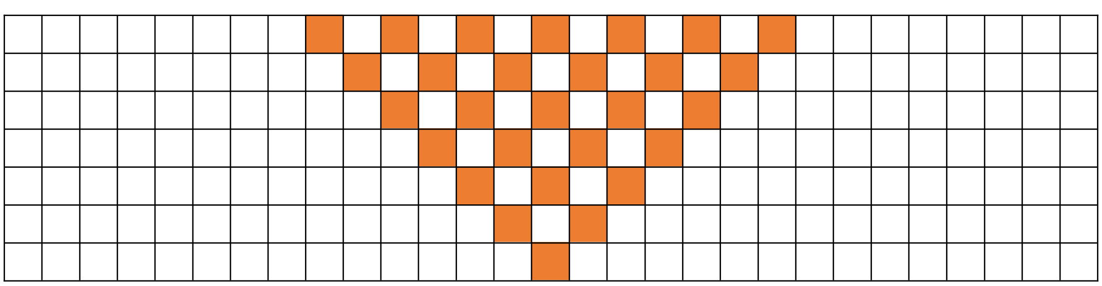
After the first step, our figure will look like this:



Now, this might look as if the evolution of the cells happened in a two-dimensional environment, but that is definitely not the case. The rows are just stacked and each of them represents a different stage of the evolution. The evolution ends just before the stage when there are no cells left. So, the final result is the following image:

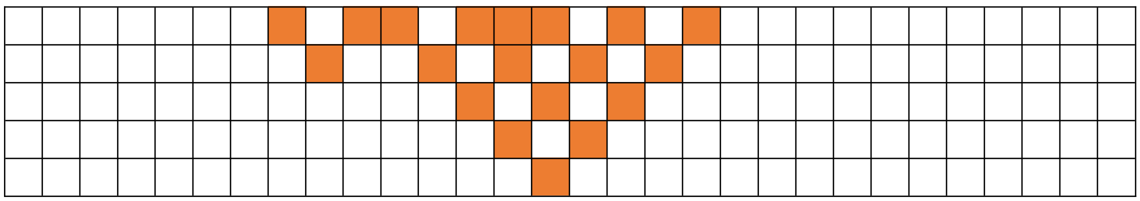


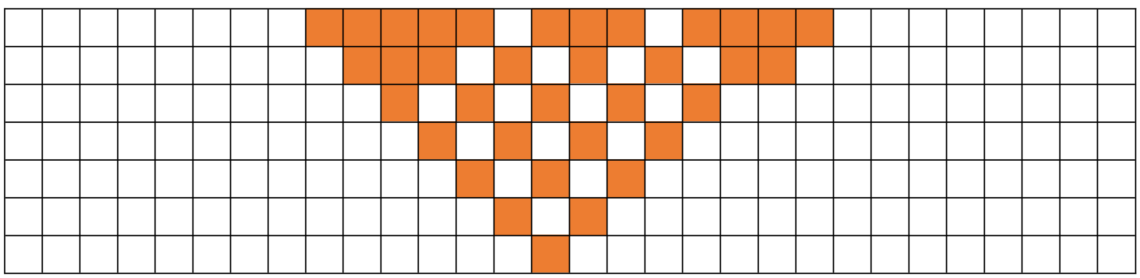
As you can see here, we started with cells that have no space between them. This is a rule that does not allow new cells that easily and the number of cells decreases constantly. In each cycle, the number of cells goes down by 2. Here we had an odd number of starting cells, so the evolution stops at 1 cell. However, if we would have an even number of cells, we would end up with two adjacent cells that will then both disappear.



In the second example, we started with cells separated by one space. This is an extremely important situation, given the properties of the rule, with this being the only time two living cells can create a new cell. Here, the rule behaves a little bit different. The number of cells decreases just by one, each pair of cells giving life to another.

Now, let's move on to some irregular first row arrangements. We've previously had regular starting shapes, and there are high chances we can see a pattern in their evolution. Irregular shapes are a bit more unpredictable and therefore can give us a better understanding of the rule.





Now, these further examples help us in generalizing the behaviour of our rule. Firstly, let us denote a dead cell with 0, a single living cell with 1, and the number of living cells in a continuous string with n. For cell arrangements that can be written as $n_1, 0, n_2, 0, n_3, 0, \dots, 0, n_k$ with $n_i \geq 3$, the next stage will look like this:

$$(n_1-2), 0, 1, 0, (n_2-2), 0, 1, 0, (n_3-2), 0, 1, 0, \dots, 0, 1, 0, (n_k-2)$$

If none of the cell strings hit a number lower than 3, then the next stage can be written as

$$(n_1-4), 0, 1, 0, (n_2-4), 0, 1, 0, (n_3-4), 0, 1, 0, \dots, 0, 1, 0, (n_k-4)$$

and so on.

If some n_i is below 3, there aren't major changes, but they still need to be mentioned:

If $n_i = 2$ we replace the $1, 0, (n_i-2), 0, 1$ block in the representation above with a $1, 0, 0, 1$ block. After that, the cells to the left of the block will never interact with the cells to the right of the block so we can split our analysis to the two separate groups and continue as before. See step 2 in the first irregular example above to see an example of this.

If $n_i = 1$ we replace the $0, 1, 0, (n_i-2), 0, 1, 0$ block in the representation above with a $0, 1, 0, 1, 0$ block. We can analogously deal with the situation when these blocks appear on the edges.

However, even now this still looks as if the generalization does not cover every path, since we are just defining a particular case. However, if we would have cells separated by more than one space, we would still get to the situations shown in the examples. So, if we have a cell arrangement that can be written as $n_1, m_1, n_2, m_2, \dots, n_{k-1}, m_{k-1}, n_k$ with each m_i representing the number of dead cells in a row, and some $m_j > 1$, then we can split this string into two, the one to the left of m_j and the one to the right of m_j and analyse these separately using the method above. There will be no interactions between these two strings. That is because the cells disappear starting from the margins and ending in the center of the string so, during the next stages, there will never be situations where two cells reach a gap of just one space to be able to interact.

Now we move on to a different rule. We will present the rule where cells are born when they have one living neighbour and a living cell dies no matter how many living neighbours it has.

We have to talk a bit about fractals, since this is exactly what are we going to find here when we start with one cell. So, what are they? The simplest explanation is the following: a fractal is a rough or fragmented geometrical shape that can be split into parts, each of which is (at least approximately) a reduced-sized copy of the whole.

Here is what we can observe when we stack together the rows that show us how the cells evolve if we start with a single living cell:

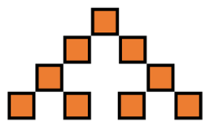


Figure 18 The evolution after four steps

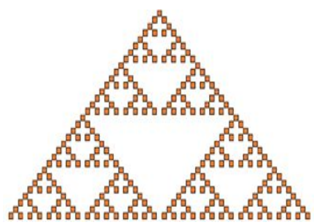


Figure 19 The evolution after 32 steps

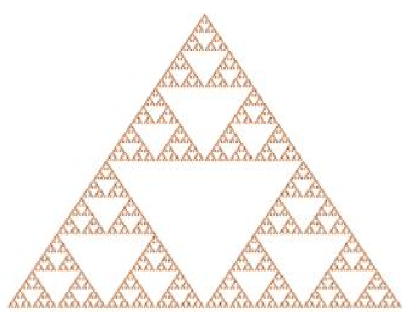


Figure 20 The evolution after 256 steps. The fractal shape is really evident now.

The shape in figure 18 is the main shape that is repeating. As you can see in the next stages, we obtain a larger and larger triangle, the process continuing infinitely. As we can observe, from step 256 onwards the triangle is very similar to Sierpiński's Triangle, which is a remarkable property.

In the final part of this section we discuss the Sierpiński Triangle and some of its properties.

The Sierpiński Triangle is a fascinating mathematical construction named after Wacław Sierpiński (14 March 1882 – 21 October 1969). He is a prominent Polish mathematician known for his contributions to set theory, number theory, and topology.

Let's delve into the mechanism behind the formation of the Sierpinski Triangle. We begin with an equilateral triangle. We then divide it into 4 smaller equilateral triangles with half the side length and remove the middle triangle. Thus, three identical triangles appear. We can now repeat this process for each of these 3 triangles, obtaining 9 even smaller triangles. We then repeat the process for these triangles and so on to infinity. This recursive process creates countless patterns of smaller triangles within a larger framework, contributing to the Sierpiński Triangle's captivating beauty.

Figure 21 The Sierpiński triangle at stage 0

Figure 22 The Sierpiński triangle after step 3

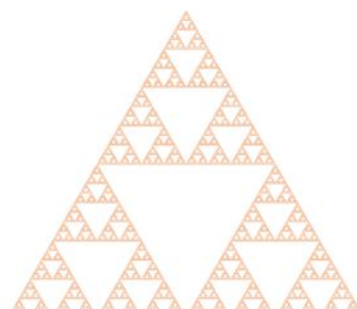


Figure 23 The Sierpiński triangle after step 5

As can be observed, this pattern becomes more and more similar to the one obtained above for the evolution of the one-dimensional rule. To make this more precise, note that (up to rescaling) the equilateral triangle in figure 21 is repeated the same number of times and in the same arrangement in the Sierpiński triangle after step n as the pixelated triangle of figure 18 is in the evolution after 2^{n+2} steps of the one-dimensional rule (7).

One particularly interesting property of the Sierpiński triangle is that its area is 0 while its perimeter is infinite. To see why, let n be the number of steps that have gone by. Then, the total perimeter of the 3^n triangles at this stage is $L \cdot 3 \cdot \left(\frac{3}{2}\right)^n$ and the total area of these triangles is $L^2 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{3}{4}\right)^n$ where L is the side length of the initial triangle. Because the process never ends, n tends to infinity. This means that the perimeter tends to infinity, because $\frac{3}{2} > 1$, while the area tends to 0, because $\frac{3}{4} < 1$.

4. CONCLUSION

To sum up, our investigation into the Game of Life on hexagons and triangles has been a rewarding journey into the world of mathematical patterns. By analysing different rules, we navigated through stable, oscillating, and gliding patterns, revealing the dynamic nature of different rules. We also looked into the one-dimensional Game of Life and found some interesting results, including structures that resembled fractals. The range of findings demonstrates the complexity and diversity of cell behaviour on different geometric shapes. Overall, our research adds to our understanding of mathematical patterns while demonstrating the Game of Life's adaptability to various tilings.

EDITING NOTES

(1) This definition of regular tiling is very restrictive. One could also say that all periodic tilings are regular, and even the Penrose tiling is regular in a certain sense.

(2) The code can be found at https://github.com/arunarjunakani/HexagonalGameOfLife. You can download it (index, scripts and css files) and run it on your computer. The lines to be modified to change the rule are in the file src.js.

(3) More precisely: because the rules only depend on the *number of living* immediate neighbours.

(4) In addition, polymino 6 in Figure 6 becomes polymino 3, both polyminoes 7 and 12 become polymino 9, and all three remain stable thereafter.

(5) In fact, except the first, all the polyminoes in figure 6 give periodic configurations.

(6) As the rule is unchanged, there is no need to edit the code. The examples of shapes given below can be tested directly on the interactive site indicated in footnote 3. Other interesting examples are shown in the video.

(7) We suggest the reader to provide a proof of this observation. It follows that, by first changing the scale on one of the axes to get equilateral triangles and then renormalizing at each step, the living cells at step 2^n correspond exactly to the upper vertices of the triangles at step n of Sierpiński's construction, and so the Sierpiński triangle then appears as the limit figure.