

# Exploring Lill's method: beyond graphical solution of polynomial equation

Année 2021-2022

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## Abstract

The aim of this article is an in-depth study of Lill's method, an ingenious graphical method of finding the roots of polynomials of any degree developed by Austrian engineer Eduard Lill and published on the *Nouvelles annales de mathématiques* in 1867 where the proof is left to the reader. Initially we analyze the original method to better understand how it works and we produce some proofs about its fundamental properties and a couple of results: we recognize a nice connection with the well known Ruffini's method for factoring polynomials and we use its geometrical properties to represent particular algebraic numbers and to give an expression for the number  $\pi$  [1]. Finally we generalize the method and by exploiting its properties we show how it allows to study the problem of inscribing regular polygons inside other regular polygons.

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## 1 Lill's method and the factorisation

### Introduction

The aim of this work is an in-depth study of Lill's method, an ingenious graphical method of finding the roots of polynomials of any degree developed by Austrian engineer Eduard Lill and published in 1867 in *Nouvelles Annales de mathématiques, Résolution graphique des équations numériques de tous*

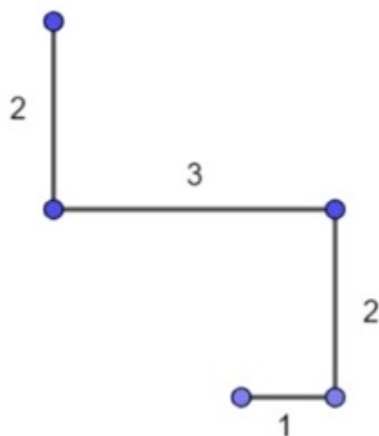
*degrés à une seule inconnue, et description d'un instrument inventé dans ce but, (1867)*, where the proof is left to the reader. Let's see first how the method actually works.

## 1.1 Lill's method

First of all we propose an example to help understand the method. Let's consider the polynomial:

$$p(x) = x^3 - 2x^2 + 3x + 2$$

We draw its path where the length of the segments correspond to the respective coefficients of the polynomial 1, -2, 3, 2



In particular, after drawing the first segment rightwards, by convention, from the end of the first segment another segment is drawn rightwards (relative to the direction established by the first segment) if the second coefficient has the same sign of the first one, leftwards if not.

The next segment is drawn rightwards if the third coefficient has the same sign of the second one, leftwards if not, and so on.

Actually this rule is slightly different from the one used by Lill. In fact we have chosen to invert the rotation of the segment with respect to Lill in order to have the positive roots above the starting segment and the negative ones below it. This choice will become clearer in the following.

The next step is to find a root of the polynomial. Let's now consider the following polynomial

$$p(x) = x^3 + 5x^2 + 7x + 3$$

whose roots are  $-1$ , double, and  $-3$ , and its path is depicted in Fig. 1

To find the roots we have to find a path, starting from  $A$ , which reflects one time on each segment (or on its own prolongation) with right angles and gets to the end of the last one.

For example, none of the paths presented in Fig. 2 is related to a root as it does not end at point  $E$ .

On the other hand, the two paths reported in Fig. 3 connect  $A$  with  $E$  therefore the length of the segments  $BF$  and  $BG$  are the roots of the polynomial  $p(x)$  (as briefly explained above they have a negative sign because they are positioned below the first segment  $AB$ ) [2].

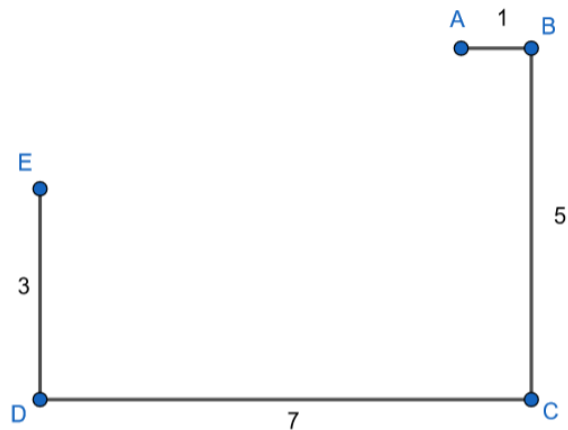


Figure 1: Path built using the polynomial  $p(x) = x^3 + 5x^2 + 7x + 3$

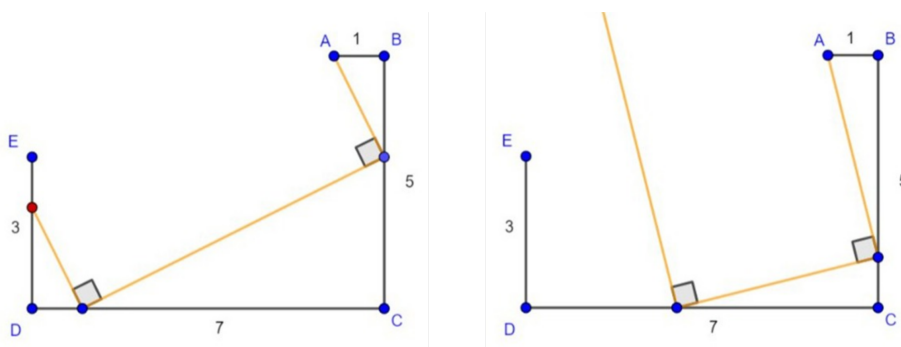


Figure 2: Both the path that start from the point A are not related to a root of the polynomial since they do not end on point E.

## 1.2 Factoring a polynomial with the Lill's method

In this section we are going to prove we can use Lill's method to factor a polynomial and that we will obtain the same factorization as the one obtained applying Ruffini's method. We know how to use the Ruffini's method to factor a polynomial, for example given

$$p(x) = x^3 - 4x^2 + 5x - 2$$

we have to

- find a zero of  $p(x)$  due to the theorem of the rational zero of a polynomial:  $x_0 = 1$ ;
- find the quotient polynomial  $q(x)$  through the euclidean division:  $q(x) = x^2 - 3x + 2$ ;
- factor  $p(x)$  thanks to Ruffini's theorem into  $(x - x_0) \cdot q(x)$

$$p(x) = (x - 1)(x^2 - 3x + 2)$$

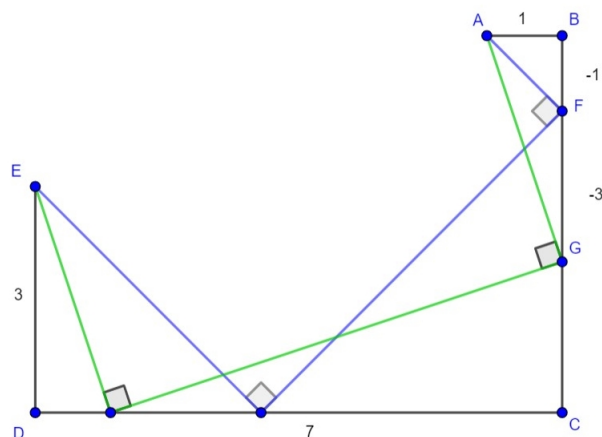


Figure 3: Two examples of path related to two roots of the polynomial  $p(x) = x^3 + 5x^2 + 7x + 3$

after, iterating the method, we find

$$p(x) = (x - 1)(x - 1)(x - 2)$$

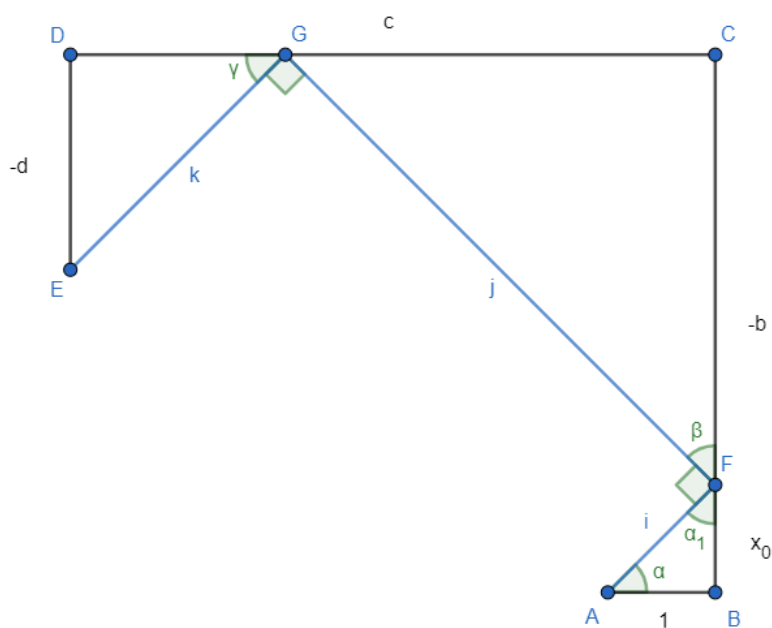
Now, before using Lill's method, we have to state two theorems:

**Theorem 1.** Given the monic polynomial  $p(x) = x^3 + bx^2 + cx + d$  with  $b < 0$ ,  $c > 0$ ,  $d < 0$  then

$$p(x) = (x - \tan \alpha) \cdot (x^2 + (b + \tan \alpha)x - d \cot \alpha)$$

where  $\alpha$  is the angle between the first segment of the path of  $p(x)$  and the first segment of the solution path.

*Proof.* Let's draw the Lill's path and the solution path of a polynomial  $p(x)$  are:



Since  $\alpha + \alpha_1 = 90^\circ$  for the theorem of the sum of the internal angles of a triangle and  $\alpha_1 + \beta = 90^\circ$  then  $\alpha = \beta$ . The same thing can be proved for the angles  $\beta$  and  $\gamma$ .

Thanks to the definition of *cosine* and *sine* we know that:

- $1 = i \cdot \cos \alpha$
- $-b = i \cdot \sin \alpha + j \cdot \cos \alpha$
- $c = j \cdot \sin \alpha + k \cdot \cos \alpha$
- $-d = k \cdot \sin \alpha$

And if we substitute these new values into the polynomial, we get:

$$p(x) = i \cdot \cos \alpha \cdot x^3 - (i \cdot \sin \alpha + j \cdot \cos \alpha) x^2 + (j \cdot \sin \alpha + k \cdot \cos \alpha) x - k \cdot \sin \alpha$$

which is equal to:

$$\begin{aligned} & i(x^3 \cos \alpha - x^2 \sin \alpha) - j(x^2 \cos \alpha - x \cdot \sin \alpha) + k(x \cdot \cos \alpha - \sin \alpha) \\ &= ix^2(x \cdot \cos \alpha - \sin \alpha) - jx(x \cdot \cos \alpha - \sin \alpha) + k(x \cdot \cos \alpha - \sin \alpha) \\ &= (x \cdot \cos \alpha - \sin \alpha)(ix^2 - jx + k) \end{aligned}$$

For the definition of *tangent*,  $\sin \alpha = \cos \alpha \cdot \tan \alpha$ .

Since  $\tan \alpha = \frac{x_0}{1} = x_0 \Rightarrow \sin \alpha = \cos \alpha \cdot x_0$ , And if we substitute that to the previous polynomial,

$$(x \cdot \cos \alpha - x_0 \cdot \cos \alpha)(ix^2 - jx + k) = \cos \alpha(x - x_0)(ix^2 - jx + k)$$

Since  $i = \frac{1}{\cos \alpha}$ ,  $j = \frac{-b - \tan \alpha}{\cos \alpha}$ ,  $k = \frac{-d}{\sin \alpha}$ , the previous polynomial is equal to:

$$\cos \alpha(x - \tan \alpha) \left( \frac{1}{\cos \alpha} x^2 + \frac{b + \tan \alpha}{\cos \alpha} x - \frac{d}{\sin \alpha} \right)$$

So  $p(x) = (x - \tan \alpha) \cdot q(x)$ , where

$$q(x) = \cos \alpha \left( \frac{1}{\cos \alpha} x^2 + \frac{b + \tan \alpha}{\cos \alpha} x - \frac{d}{\sin \alpha} \right) = x^2 + (b + \tan \alpha)x - d \cot \alpha$$

□

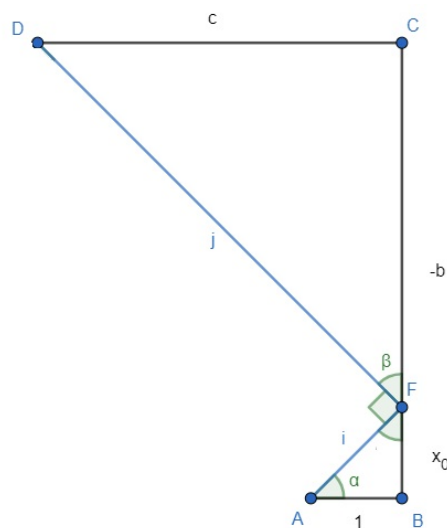
We can also see that  $\frac{1}{\cos \alpha}$ ,  $\frac{b + \tan \alpha}{\cos \alpha}$ ,  $-\frac{d}{\sin \alpha}$  are the lengths of the segments of the solution path, which becomes another polynomial path  $\frac{1}{\cos \alpha} x^2 + \frac{b + \tan \alpha}{\cos \alpha} x - \frac{d}{\sin \alpha}$  rotated by the  $\alpha$  angle.

**Theorem 2.** Given the monic polynomial  $p(x) = x^2 - bx + c$  with  $b > 0$ ,  $c < 0$  then

$$p(x) = (x - \tan \alpha) \cdot (x - c \cot \alpha)$$

where  $\alpha$  is the angle between the first segment of the path of  $p(x)$  and the first segment of the solution path.

*Proof.* In a similar way to the previous theorem we draw



and we observe that:

- $1 = i \cdot \cos \alpha$
- $-b = i \cdot \sin \alpha + j \cdot \cos \alpha$
- $c = j \cdot \sin \alpha$

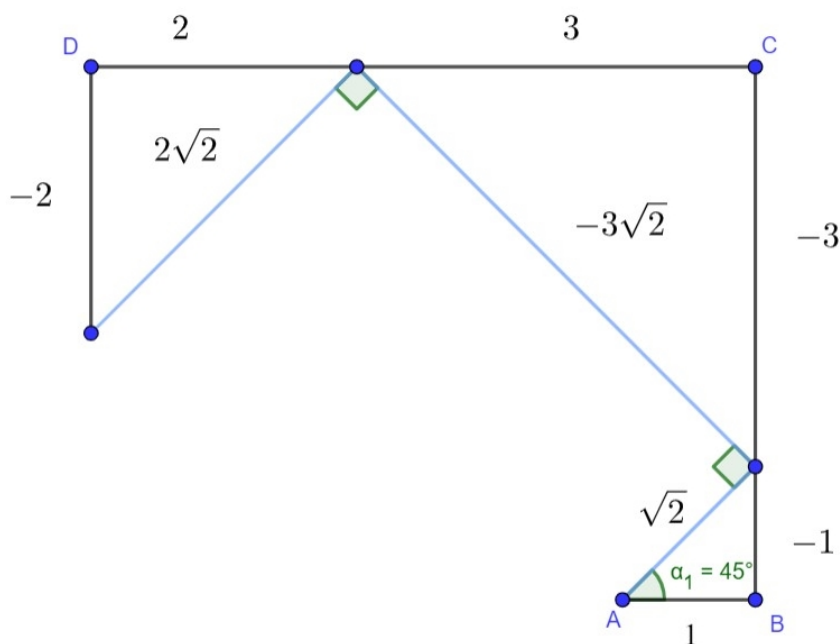
therefore

$$\begin{aligned}
 p(x) &= i \cdot \cos \alpha \cdot x^2 - (i \cdot \sin \alpha + j \cdot \cos \alpha) x + j \cdot \sin \alpha \\
 &= (x \cdot \cos \alpha - \sin \alpha)(ix - j) \\
 &= \cos \alpha (x - x_0)(ix - j) \\
 &= \cos \alpha (x - \tan \alpha) \left( \frac{1}{\cos \alpha} x - \frac{c}{\sin \alpha} \right) \\
 &= (x - \tan \alpha)(x - c \cot \alpha)
 \end{aligned}$$

□

We can also see that  $\frac{1}{\cos \alpha}$ ,  $-\frac{c}{\sin \alpha}$  are the lengths of the segments of the solution path, which becomes another polynomial path  $\frac{1}{\cos \alpha} x - \frac{c}{\sin \alpha}$  rotated by the  $\alpha$  angle.

Now we can factor the polynomial  $p(x) = x^3 - 4x^2 + 5x - 2$  also with Lill's method. First of all we draw the path and a solution path obtained with  $\alpha_1 = 45^\circ$ :



then we use the first theorem above and we get

$$\begin{aligned}
 p(x) &= (x - \tan \alpha) \cdot (x^2 + (b + \tan \alpha)x - d \cot \alpha) \\
 &= \left(x - \tan \frac{\pi}{4}\right) \cdot \left(x^2 + \left(b + \tan \frac{\pi}{4}\right)x - d \cot \frac{\pi}{4}\right) \\
 &= (x - 1)(x^2 - 3x + 2)
 \end{aligned}$$

We observe that

$$\frac{1}{\cos \alpha} x^2 + \frac{b + \tan \alpha}{\cos \alpha} x - \frac{d}{\sin \alpha} = \sqrt{2}x^2 - 3\sqrt{2}x + 2\sqrt{2}$$

is a new path rotated by the  $\alpha_1 = 45^\circ$  angle and dilated by  $\sqrt{2}$ .

Iterating the process by drawing the solution path related to the first solution path and thanks to the second theorem above we get

$$x^2 - 3x + 2 = (x - \tan \alpha_2)(x - c \cot \alpha_2) = (x - 1)(x - 2)$$

We observe that

$$\frac{1}{\cos \alpha} x - \frac{c}{\sin \alpha} = \sqrt{2}x - 2\sqrt{2}$$

is a new path rotated by the  $\alpha_2 = 45^\circ$  angle and dilated of  $\sqrt{2}$ .

By combining the two rotodilatations we get  $x - 2$  in fact

$$p(x) = \frac{\sqrt{2}}{2}(x - 1)(\sqrt{2}x^2 - 3\sqrt{2}x + 2\sqrt{2}) = \frac{\sqrt{2}}{2}(x - 1)\sqrt{2}(x^2 - 3x + 2)$$

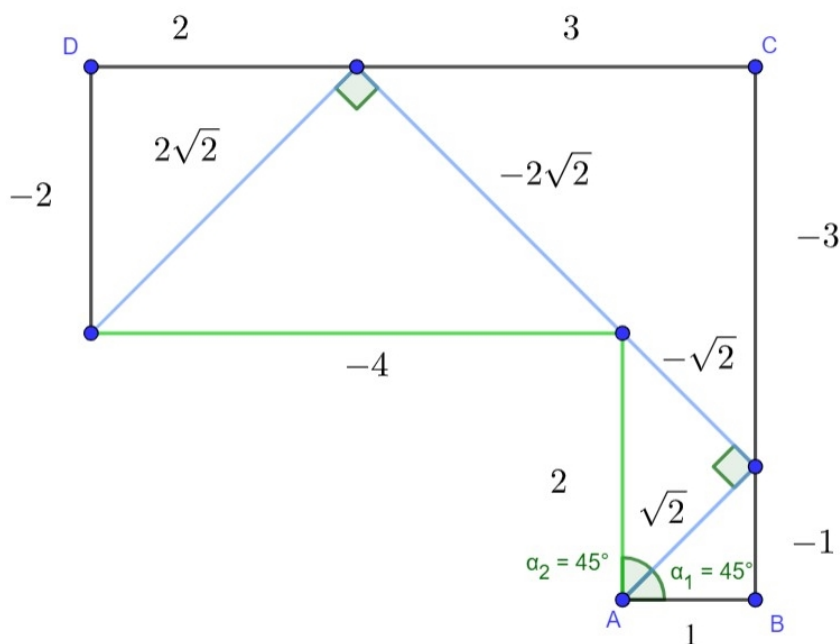


Figure 4: the external path represents the starting polynomial  $p(x) = x^3 - 4x^2 + 5x - 2$ . Each internal path instead is related to a factorization step that ends up being  $p(x) = (x - 1)^2(x - 2)$

we can factor it again into

$$\frac{\sqrt{2}}{2}(x - 1)\sqrt{2} \cdot \frac{\sqrt{2}}{2}(x - 1)(\sqrt{2}x - 2\sqrt{2}) = (x - 1)^2(x - 2)$$

in conclusion we got the same factorisation with both methods.

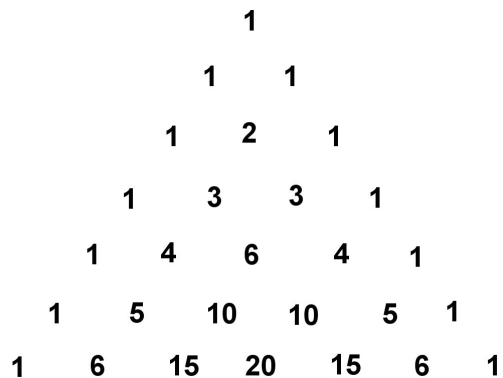
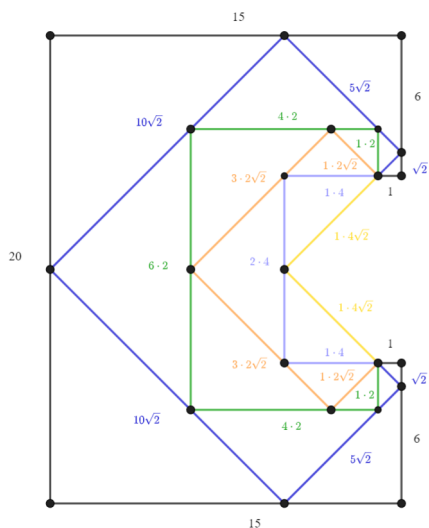
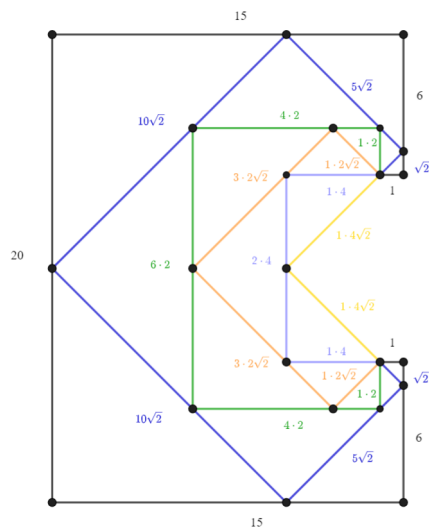
### 1.3 Lill's method and the Pascal's triangle

We can also consider the specific case of a polynomial  $p(x)$  which is the result of the power  $(x - a)^n$  with  $n \in \mathbb{N}$ , for example

$$(x - 1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$$

We can draw its polynomial path and then, inside, the solution path that is the lower power. If we continue to draw the paths we will get the nice symmetrical construction of the following figure in which the lengths of the segments of the paths are multiples of the coefficients of Pascal's triangle (known in Italy as Tartaglia's triangle).





As we have seen above

$$(x - 1)^6 = (x - \tan 45^\circ) q_5(x)$$

where  $q_5(x) = (x - 1)^5$  and the path

$$\frac{q_5(x)}{\cos 45^\circ} = \sqrt{2}(x - 1)^5$$

corresponds to the solution path of  $(x - 1)^6$ .

Subsequently

$$(x - 1)^5 = (x - \tan 45^\circ) q_4(x)$$

where  $q_4(x) = (x - 1)^4$  and the path

$$\frac{q_4(x)}{\cos 45^\circ \cdot \cos 45^\circ} = 2(x - 1)^4$$

corresponds to the solution path of  $(x - 1)^5$ .

Subsequently

$$(x - 1)^4 = (x - \tan 45^\circ) q_3(x)$$

where  $q_3(x) = (x - 1)^3$  and the path

$$\frac{q_3(x)}{\cos 45^\circ \cdot \cos 45^\circ \cdot \cos 45^\circ} = 2\sqrt{2}(x - 1)^3$$

corresponds to the solution path of  $(x - 1)^4$ .

In the same way we find out the solution path  $4(x - 1)^2$  and at the end  $4\sqrt{2}(x - 1)$ .

## 2 Generalization

### 2.1 Definitions

Starting from the analysis of the method we are going to give some rigorous definitions. Then we are going to state and prove the theorem that describes the reason why the method works that way.

**Definition 1.** Given a polynomial  $p(x)$  of degree  $n \in \mathbb{N}_0$

$$p(x) := a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

and an angle  $0 < \varphi < \pi$  a **Polynomial path**  $P_\varphi$  in the complex plane is defined by  $q(e^{i\varphi})$  where:

$$q(x) := \frac{p(x)}{a_n x^n} = 1 + \frac{a_{n-1}}{a_n} x^{-1} + \dots + \frac{a_1}{a_n} x^{1-n} + \frac{a_0}{a_n} x^{-n}$$

In the following we will consider w.l.o.g. a monic  $p(x)$  ( $a_n = 1$ ) and in this case its polynomial path will be

$$q(e^{i\varphi}) = \underbrace{1}_{v_0} + \underbrace{a_{n-1} e^{-i\varphi}}_{v_1} + \underbrace{a_{n-2} e^{-2i\varphi}}_{v_2} + \dots + \underbrace{a_1 e^{(1-n)i\varphi}}_{v_{n-1}} + \underbrace{a_0 e^{-ni\varphi}}_{v_n}$$

Where each term has to be intended as a vector on the complex plane.

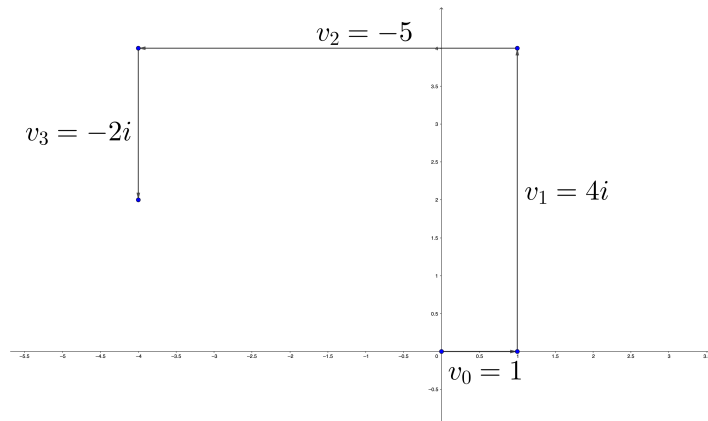
**Example 1.** Given the angle  $\varphi = \frac{\pi}{2}$  and the polynomial  $p(x) = x^3 - 4x^2 + 5x^2 - 2$ .

Let's draw its Polynomial Path  $P_{\frac{\pi}{2}}$ . First of all let's calculate  $q(x)$ :

$$q(x) = \frac{p(x)}{x^5} = 1 - 4x^{-1} + 5x^{-2} - 2x^{-3}$$

then

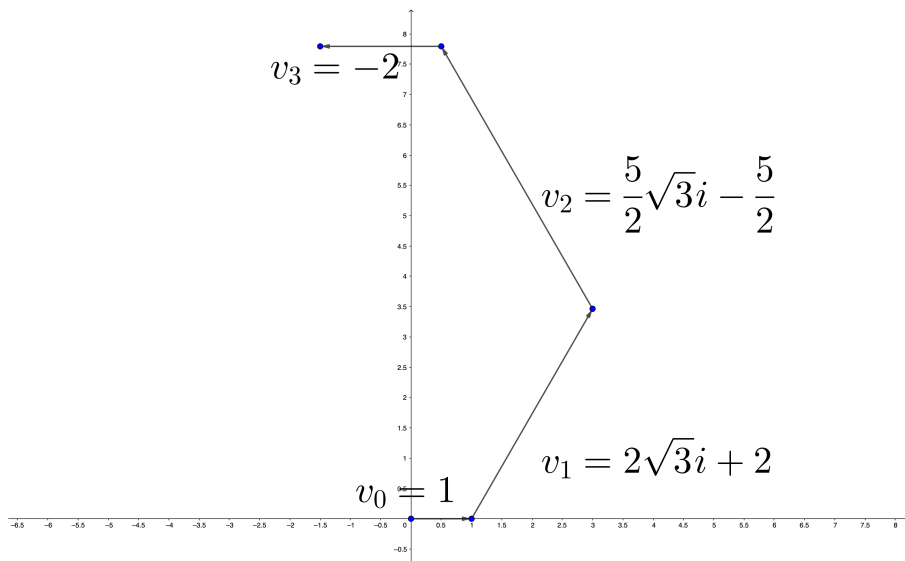
$$q\left(e^{i\frac{\pi}{2}}\right) = \underbrace{1}_{v_0} - \underbrace{4e^{-\frac{\pi}{2}i}}_{v_1} + \underbrace{5e^{-\pi i}}_{v_2} - \underbrace{2e^{-\frac{3}{2}\pi i}}_{v_3} = \underbrace{1}_{v_0} + \underbrace{+4i}_{v_1} - \underbrace{5}_{v_2} - \underbrace{-2i}_{v_3}$$



Let's now use a different angle such as  $\varphi = \frac{2}{3}\pi$ , then:

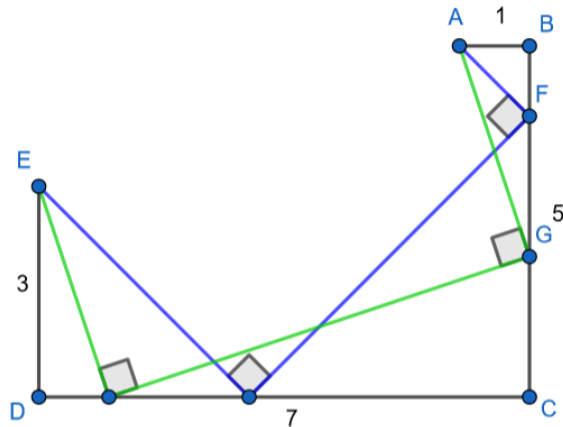
$$q\left(e^{i\frac{2}{3}\pi}\right) = \underbrace{1}_{v_0} - \underbrace{4e^{-\frac{2}{3}\pi i}}_{v_1} + \underbrace{5e^{-\frac{4}{3}\pi i}}_{v_2} - \underbrace{2e^{-2\pi i}}_{v_3} = \underbrace{1}_{v_0} + \underbrace{2\sqrt{3}i + 2}_{v_1} + \underbrace{\frac{5}{2}\sqrt{3}i - \frac{5}{2}}_{v_2} - \underbrace{2}_{v_3}$$

And its polynomial path  $P_{\frac{2}{3}\pi}$  becomes:



**Definition 2.** Given a polynomial path  $P_\varphi$  a **Reflection path**  $R_\varphi$  is a set of consecutive line segments that intersect one and only one time each segment of  $P_\varphi$  or its prolongation with a reflection of angle  $\varphi$ .

Here is an example:

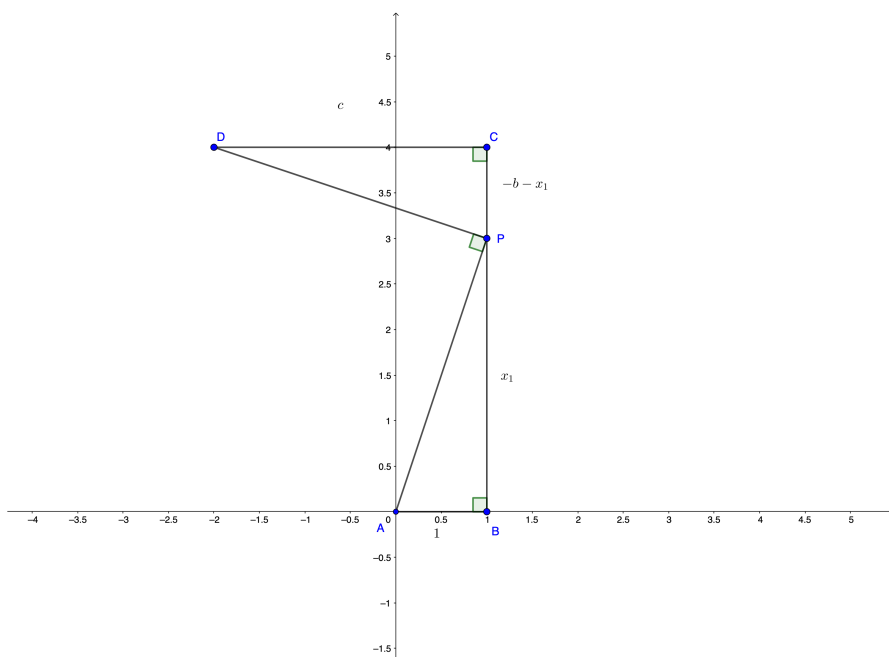


## 2.2 Theorems

**Theorem 3.** Given a polynomial path  $P_\varphi$ , if the the last end-point of a reflection path  $R_\varphi$  coincide with the last end-point of  $P_\varphi$  then the distance (or its opposite) from the second end-point of  $P_\varphi$  to the second end-point of  $R_\varphi$  is a root of the polynomial  $p(x)$ , and we call this  $R_\varphi$  **Solution Path**  $S_\varphi$  [3].

*Proof.* We are going to prove the theorem starting from the polynomials of degree 2: in this case we would have 4 different polynomial path depending on the sign of  $b$  and  $c$  with  $a = 1$ . However we can reduce them to two cases due to the symmetry with respect to the  $x$ -axis. Let's consider a polynomial path  $P_{\frac{\pi}{2}}$  of a polynomial of degree 2:

$$p(x) = ax^2 + bx + c \quad \text{with } a = 1, b < 0, c > 0$$



Let's  $\overline{BP} = x_1$ ,  $\overline{PC} = -b - x_1$ , and  $\overline{CD} = c$ .

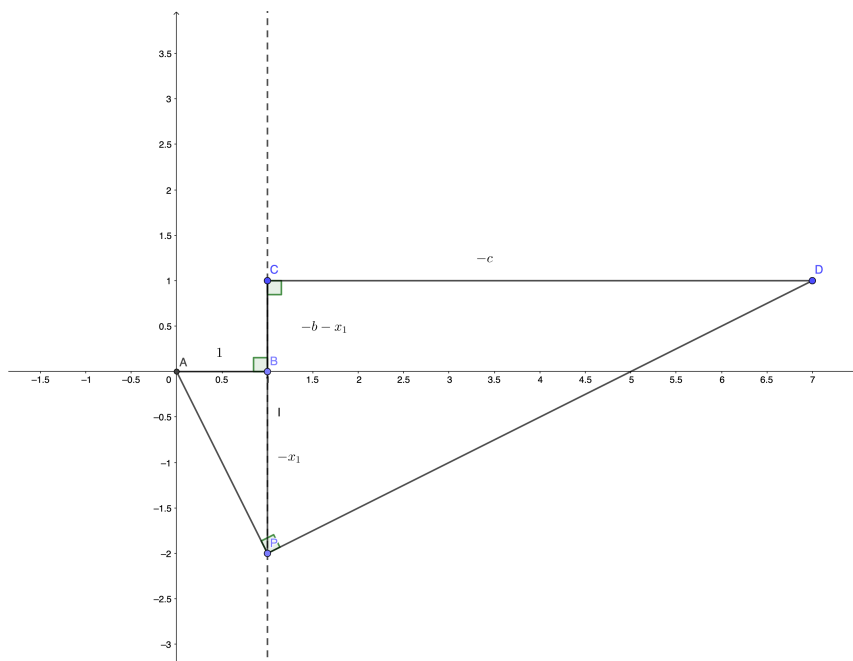
Let's consider the triangles  $(ABP)$  and  $(DCP)$ , they have:

$$\left. \begin{array}{l} \widehat{ABP} \cong \widehat{DCP} \cong \frac{\pi}{2} \text{ for construction} \\ \widehat{DPC} \cong \widehat{PAB} \text{ because they are} \\ \text{complementary to the same angle } \widehat{APB} \end{array} \right\} \Rightarrow \text{for the similarity criterion} \Rightarrow \text{triangle } (ABP) \sim \text{triangle } (DCP)$$

$$\frac{\overline{PB}}{\overline{DC}} = \frac{\overline{AB}}{\overline{PC}} \Rightarrow \frac{x_1}{c} = \frac{1}{-b-x_1} \Rightarrow -(b+x_1)x_1 = c \Rightarrow x_1^2 + bx_1 + c = 0$$

Then  $x_1$  is a root of the polynomial  $p(x)$

Let's now consider the case  $b < 0$  and  $c > 0$ , then  $P_{\frac{\pi}{2}}$  becomes:



Let's  $\overline{BP} = -x_1$ ,  $\overline{PC} = -b - x_1$ , and  $\overline{CD} = -c$ .

Let's consider the triangles  $(ABP)$  and  $(DCP)$ , they have:

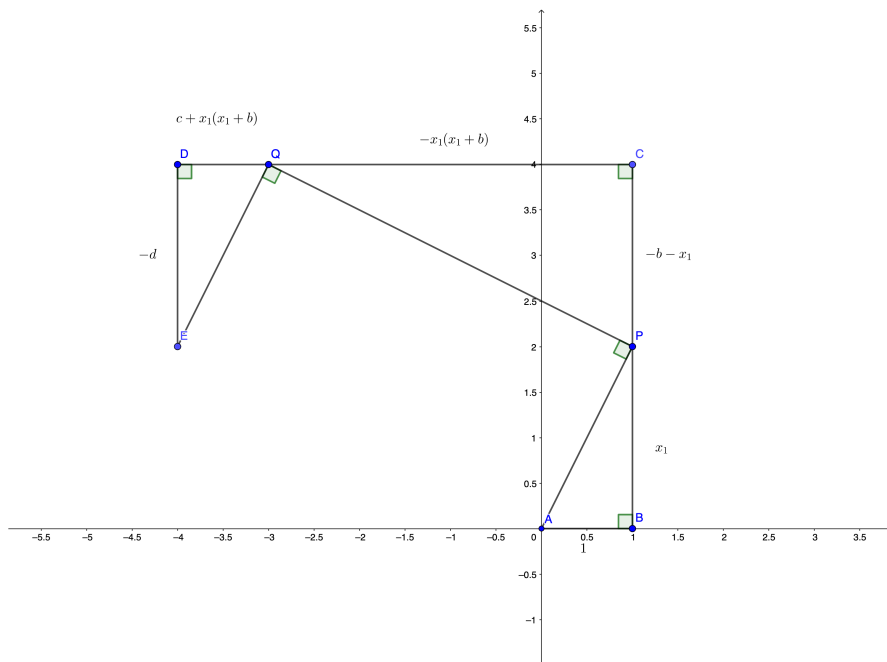
$$\left. \begin{array}{l} \widehat{ABP} \cong \widehat{DCP} \cong \widehat{R} \text{ for construction} \\ \widehat{DPC} \cong \widehat{PAB} \text{ because they are} \\ \text{complementary to the same angle } \widehat{APB} \end{array} \right\} \Rightarrow \text{for the similarity criterion} \Rightarrow \text{triangle } (ABP) \sim \text{triangle } (DCP)$$

$$\frac{\overline{PB}}{\overline{DC}} = \frac{\overline{AB}}{\overline{PC}} \Rightarrow -(b+x_1)x_1 = c \Rightarrow \frac{-x_1}{-c} = \frac{1}{-b-x_1} \Rightarrow x_1^2 + bx_1 + c = 0$$

Then  $x_1$  is a root of the polynomial  $p(x)$

Let's now consider a polynomial path  $P_{\frac{\pi}{2}}$  of a polynomial of degree 3, similarly to what we said before we would have 8 different Polynomial path that we can reduce to 4 but due to the generality of the reasoning we will use only one of them:

$$q(x) = ax^3 + bx^2 + cx + d \quad \text{with } a = 1, b < 0, c > 0, d < 0$$



Let's  $\overline{BP} = x_1, \overline{PC} = -b - x_1, \overline{CD} = c, \overline{DE} = -d$

Let's consider the triangles  $(ABP)$  and  $(QCP)$ , they have:

$$\left. \begin{array}{l} \widehat{ABP} \cong \widehat{QCP} \cong \widehat{R} \text{ for construction} \\ \widehat{QPC} \cong \widehat{PAB} \text{ because they are} \\ \text{complementary to the same angle } \widehat{APB} \end{array} \right\} \Rightarrow \text{triangle } (ABP) \sim \text{triangle } (QCP) \Rightarrow \text{for the similarity criterion}$$

$$\text{So } \frac{\overline{PB}}{\overline{QC}} = \frac{\overline{AB}}{\overline{PC}} \Rightarrow \frac{x_1}{\overline{QC}} = \frac{1}{-b-x_1} \Rightarrow \overline{QC} = -x_1(x_1 + b)$$

Then  $\overline{DQ} = c + x_1(x_1 + b)$

Let's consider the triangles  $(QDE)$  and  $(QCP)$ , they have:

$$\left. \begin{array}{l} \widehat{QDE} \cong \widehat{QCP} \cong \widehat{R} \text{ for construction} \\ \widehat{QPC} \cong \widehat{EQD} \text{ because they are} \\ \text{complementary to the same angle } \widehat{CQP} \end{array} \right\} \Rightarrow \text{triangle } (QDE) \sim \text{triangle } (QCP) \Rightarrow \text{for the first similarity criterion}$$

$$\text{So } \frac{\overline{PC}}{\overline{DQ}} = \frac{\overline{CQ}}{\overline{DE}} \Rightarrow$$

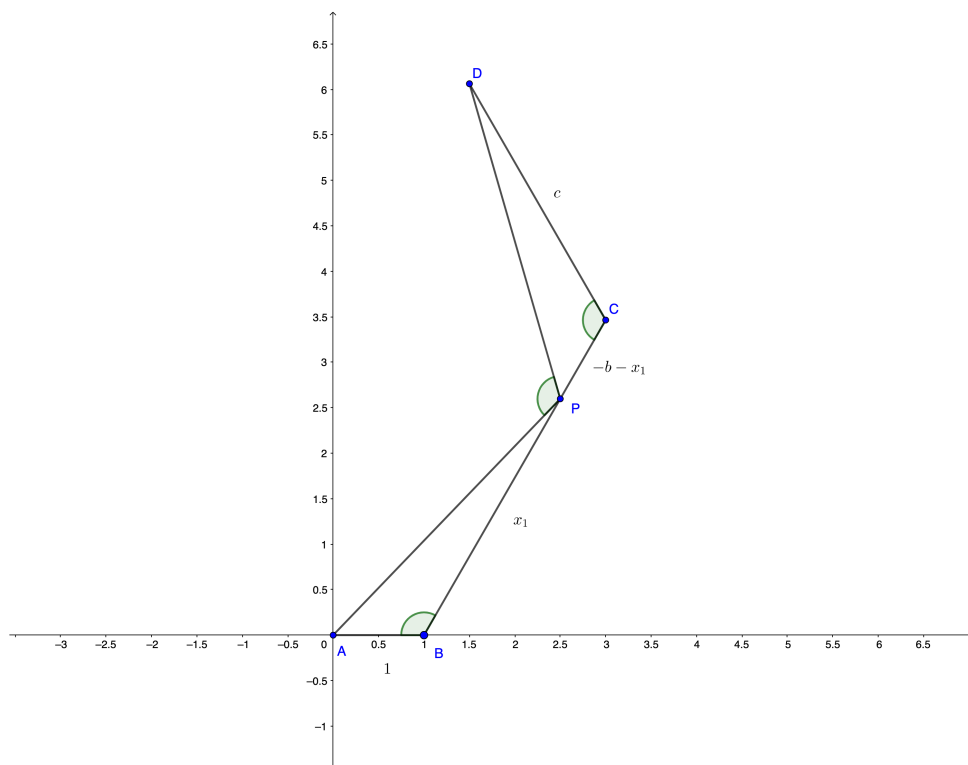
$$\frac{-b-x_1}{c+x_1(x_1+b)} = \frac{-x_1(x_1+b)}{-d} \Rightarrow d(b+x_1) = -x_1(b+x_1)(c+x_1(x_1+b)) \Rightarrow$$

$$d = -x_1c - x_1^2(x_1+b) \Rightarrow x_1^3 + bx_1^2 + cx_1 + d = 0$$

Then  $x_1$  is a root of the polynomial  $p(x)$

Let's consider a polynomial path  $P_\varphi$  of a polynomial of degree 2 with concordant roots :

$$p(x) = ax^2 + bx + c \quad \text{with } a = 1, b < 0, c > 0$$



Let's  $\overline{BP} = x_1$ ,  $\overline{PC} = -b - x_1$ , and  $\overline{CD} = c$ .

Let's consider the triangles  $(ABP)$  and  $(DCP)$ , they have:

$$\left. \begin{array}{l} \widehat{ABP} \cong \widehat{DCP} \text{ for construction} \\ \widehat{DPC} \cong \widehat{PAB} \text{ because of the} \\ \text{sum of the internal angles of triangle} \end{array} \right\} \Rightarrow \text{triangle } (ABP) \sim \text{triangle } (DCP) \Rightarrow \text{for the first similarity criterion}$$

$$\frac{\overline{PB}}{\overline{DC}} = \frac{\overline{AB}}{\overline{PC}} \Rightarrow \frac{x_1}{c} = \frac{1}{-b - x_1} \Rightarrow -(b + x_1)x_1 = c \Rightarrow x_1^2 + bx_1 + c = 0$$

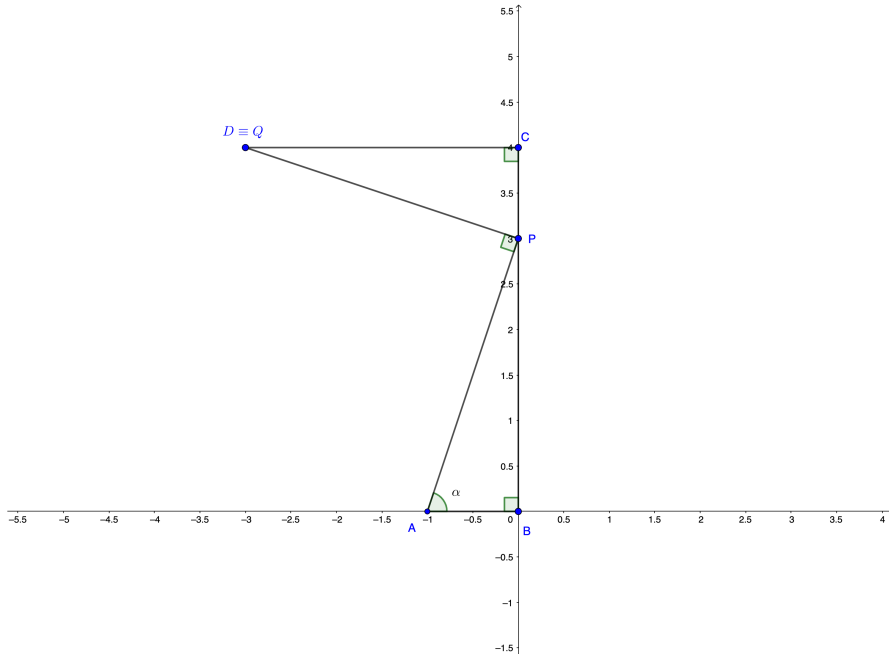
Then  $x_1$  is a root of the polynomial  $p(x)$

□

We can also use the tangent of the angle  $\alpha$  between the  $x$ -axis and the first segment of the Reflection Path to prove this theorem:

*Proof.* Given a  $P_{\frac{\pi}{2}}$  of a polynomial

$$p(x) = ax^2 + bx + c \quad \text{with } a = 1, b < 0, c > 0$$



Set the cartesian plane on the first vertex of  $P_{\frac{\pi}{2}}$  with the first segment on the  $x$ -axis and the second segment on the  $y$ -axis. Let  $P$  be a point on the second segment of  $P_{\frac{\pi}{2}}$  of  $p(x)$ . The straight line  $r_{AP}$  passing through  $A$  and  $P$  form an angle  $\alpha$  with the  $x$ -axis, being  $\overline{AB} = 1$  due to the definition of Polynomial Path the equation of  $r_{AP}$  is:

$$r_{AP} : y = \tan \alpha x + \tan \alpha$$

In order to draw the second segment of  $R_{\frac{\pi}{2}}$  of  $p(x)$  let's find the straight line perpendicular to  $r_{AP}$ :

$$m_{r_{\perp}} = -\frac{1}{m_{r_{AP}}} = -\frac{1}{\tan \alpha}$$

$$r_{\perp} : y = -\frac{1}{\tan \alpha}x + \tan \alpha$$

Let  $Q$  be  $r_{\perp} \cap r_{DC}$  with  $r_{DC} : y = -b$

$$\begin{cases} y = -b \\ y = -\frac{1}{\tan \alpha}x + \tan \alpha \end{cases} \implies -\frac{1}{\tan \alpha}x + \tan \alpha = -b \implies x_Q = \tan \alpha(\tan \alpha + b)$$

$$\tan^2 \alpha + b \tan \alpha - x_Q = 0$$

$$\tan \alpha = \frac{-b \pm \sqrt{b^2 + 4x_Q}}{2}$$

Being  $a = 1$   $p(x)$  roots are:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

So  $\tan \alpha$  is a root of  $p(x)$  only for  $x_Q = -c$  and that is true only when  $Q \equiv D$ . That said if  $Q \equiv D$  so the free end-points of  $P_{\frac{\pi}{2}}$  and  $R_{\frac{\pi}{2}}$  coincide then  $\tan \alpha$  is a root of the polynomial  $p(x)$ .  $\square$



### 3 Polygons and Polynomial Paths

We are now going to study the relationship between the Polynomial Path, and therefore the polynomials, and the regular polygons. Then we are going to analyse the regular polygons inscribable in other regular polygons.

#### 3.1 Definitions

**Definition 3.** The regular  $m$ -agon can be obtained as the Polynomial Path with angle  $\varphi_m = \frac{m-2}{m}\pi$  (amplitude of the internal angle) and

$$p_m(x) := \frac{x^m - (-1)^m}{x + 1} = \sum_{j=1}^m (-1)^{(j+1)} x^{m-j} = x^{m-1} - x^{m-2} + x^{m-3} - \dots$$

and

$$q_m(x) = \frac{p_m(x)}{x^{m-1}} = \sum_{j=1}^m (-1)^{(j+1)} x^{1-j} = \sum_{l=0}^{m-1} \left(-\frac{1}{x}\right)^l \quad \text{with } l = j - 1 \quad (1)$$

**Example 2.** Given  $m = 6$ , then

$$p_6(x) := \frac{x^6 - 1}{x + 1} = x^5 - x^4 + x^3 - x^2 + x - 1 \quad \varphi_6 = \frac{6-2}{6}\pi = \frac{2}{3}\pi$$

$$q_6(x) = 1 - x^{-1} + x^{-2} - x^{-3} + x^{-4} - x^{-5}$$

its evaluation in  $e^{i\frac{2}{3}\pi}$ :

$$q_6(e^{i\frac{2}{3}\pi}) = 1 - e^{-i\frac{2}{3}\pi} + e^{-i\frac{4}{3}\pi} - e^{-i2\pi} + e^{-i\frac{8}{3}\pi} - e^{-i\frac{8}{3}\pi}$$

and the consequent Polynomial Path is:

#### 3.2 Theorems

**Theorem 4.** The Polynomial Path given by  $q_m(e^{i\varphi_m})$  is closed.

*Proof.* Since  $q_m(e^{i\varphi_m})$  results in the sum of the unit complex roots its value is always zero therefore the consequent polynomial path is always close.  $\square$

**Theorem 5.** The regular  $m$ -agon can be obtained as the Polynomial Path with angle  $\varphi_{m \cdot k} = \frac{mk-2}{mk}\pi$  and the polynomial:

$$p'_m(x) := \frac{p_{m \cdot k}(x)}{p_k(x)}$$

So we will use the notation:

$$(p_m(x); \varphi_m) \cong (p'_m(x); \varphi_{m \cdot k})$$

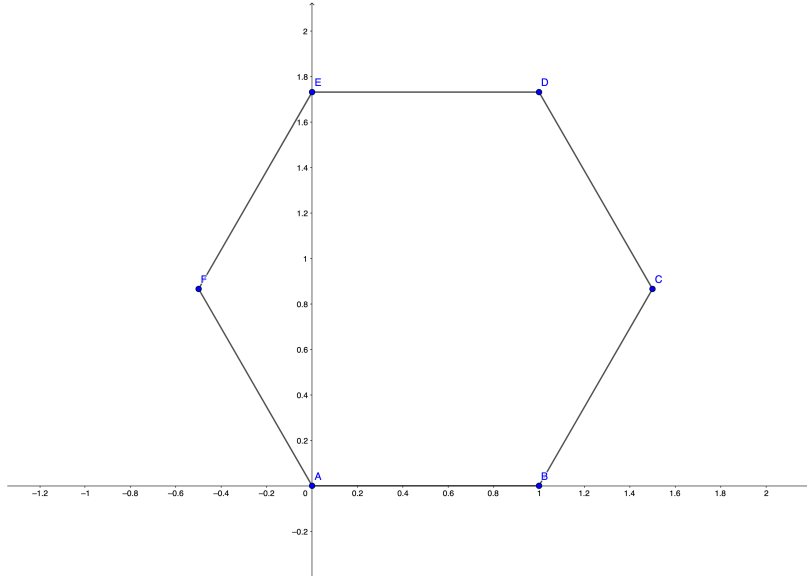


Figure 5: Polynomial Path of  $p_6(x)$

*Proof.*

$$p'_m(x) := \frac{p_{m,k}(x)}{p_k(x)} = \frac{\frac{x^{mk} - (-1)^{mk}}{x+1}}{\frac{x^k - (-1)^k}{x+1}} = \frac{x^{mk} - (-1)^{mk}}{x^k - (-1)^k} =$$

$$= \begin{cases} x^{(m-1)k} - x^{(m-2)k} + \dots = \sum_{i=1}^m (-1)^{i+1} x^{(m-i)k}, & k \text{ odd} \in \mathbb{N}_0 \\ x^{(m-1)k} + x^{(m-2)k} + \dots = \sum_{i=1}^m x^{(m-i)k}, & k \text{ even} \in \mathbb{N}_0 \end{cases} \quad (2)$$

Then

$$q'_m(x) = \frac{p'_m(x)}{x^{(m-1)k}} =$$

$$= \begin{cases} \frac{\sum_{i=1}^m (-1)^{i+1} x^{(m-i)k}}{x^{(m-1)k}} = \sum_{i=1}^m (-1)^{i+1} x^{k(1-i)} = \sum_{l=0}^{m-1} \left(-\frac{1}{x^k}\right)^l \quad (\text{with } l = i-1), & k \text{ odd} \in \mathbb{N}_0 \\ \frac{\sum_{i=1}^m x^{(m-i)k}}{x^{(m-1)k}} = \sum_{i=1}^m x^{(1-i)k} = \sum_{l=0}^{m-1} \left(\frac{1}{x^k}\right)^l \quad (\text{with } l = i-1), & k \text{ even} \in \mathbb{N}_0 \end{cases}$$

From the equation 1 in the definition 3 we have that

$$q_m(x) = \sum_{l=0}^{m-1} \left(-\frac{1}{x}\right)^l$$

let's now analyse both the cases with  $k \in \mathbb{N}_0$  *odd* or *even*:

- If  $k$  is odd

$$q'(e^{i\varphi_{mk}}) = \sum_{l=0}^{m-1} \left(-\frac{1}{(e^{i\varphi_{mk}})^k}\right)^l \quad \text{and} \quad q(e^{i\varphi_m}) = \sum_{l=0}^{m-1} \left(-\frac{1}{e^{i\varphi_m}}\right)^l$$

So, in order to have the same polynomial path:

$$\begin{aligned}
 & -\frac{1}{(e^{i\varphi_{mk}})^k} = -\frac{1}{e^{i\varphi_m}} \\
 \Rightarrow & e^{ki\varphi_{mk}} = e^{i\varphi_m} \\
 \Rightarrow & k\frac{mk-2}{mk}\pi = \frac{m-2}{m}\pi + 2\lambda\pi \quad (\lambda \in \mathbb{Z}) \\
 \Rightarrow & k\pi - \frac{2}{m}\pi = \pi - \frac{2}{m}\pi + 2\lambda\pi \\
 \Rightarrow & k\pi = \pi + 2\lambda\pi
 \end{aligned}$$

Being  $k$  odd this equation is verified for each value of  $k$

- If  $k$  is even:

$$q^l(e^{i\varphi_{mk}}) = \sum_{l=0}^{m-1} \left( \frac{1}{(e^{i\varphi_{mk}})^k} \right)^l \quad \text{and} \quad q(e^{i\varphi_m}) = \sum_{l=0}^{m-1} \left( -\frac{1}{e^{i\varphi_m}} \right)^l$$

So, in order to have the same polynomial path:

$$\begin{aligned}
 & \frac{1}{(e^{i\varphi_{mk}})^k} = -\frac{1}{e^{i\varphi_m}} \\
 \Rightarrow & e^{ki\varphi_{mk}} = -e^{i\varphi_m} \\
 \Rightarrow & e^{ki\varphi_{mk}} = e^{i\pi} e^{i\varphi_m} \\
 \Rightarrow & k\frac{mk-2}{mk}\pi = \pi + \frac{m-2}{m}\pi + 2\lambda\pi \quad (\lambda \in \mathbb{Z}) \\
 \Rightarrow & k\pi - \frac{2}{m}\pi = 2\pi - \frac{2}{m}\pi + 2\lambda\pi \\
 \Rightarrow & k\pi = 2\pi + \lambda\pi
 \end{aligned}$$

Being  $k$  even this equation is verified for each value of  $k$

□

### 3.3 Inscribed polygons

We noticed that there is a relationship between the factorisation of the polynomial and the polygons inscribable inside the initial Polynomial Path. Let's look at an example:

**Example 3.** Let's consider the polynomial that generates a regular dodecagon:

$$p_{12}(x) = \frac{x^{12}-1}{x+1} = x^{11} - x^{10} + x^9 - x^8 + x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1 \quad \varphi_{12} = \frac{5}{6}\pi$$

and its factorisation in the Real numbers:

$$p_{12}(x) = (x-1)(x^2-x+1)(x^2+1)(x^2+x+1)(x^4-x^2+1)$$

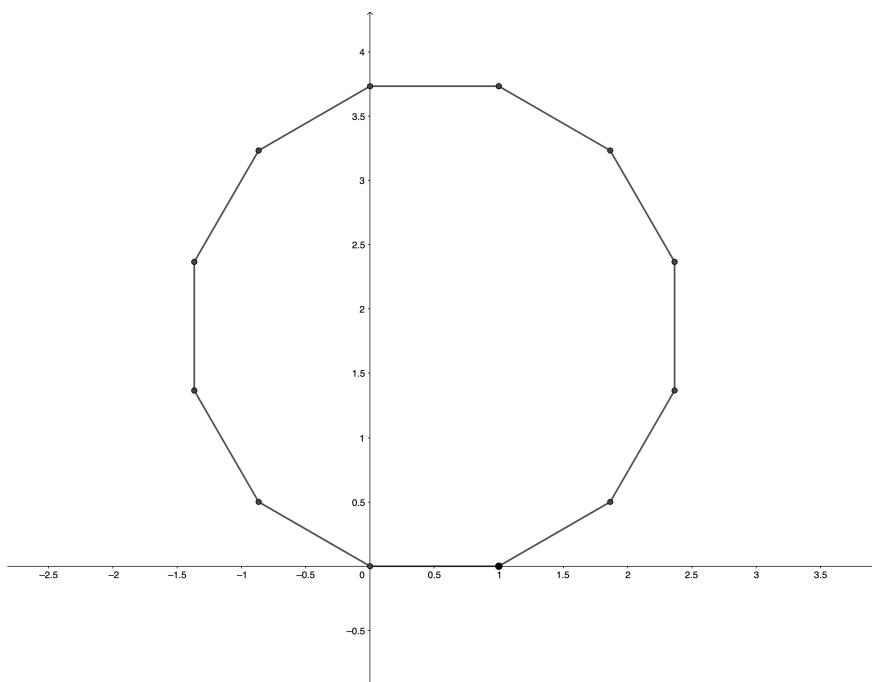


Figure 6: Polynomial Path of  $p_{12}(x)$

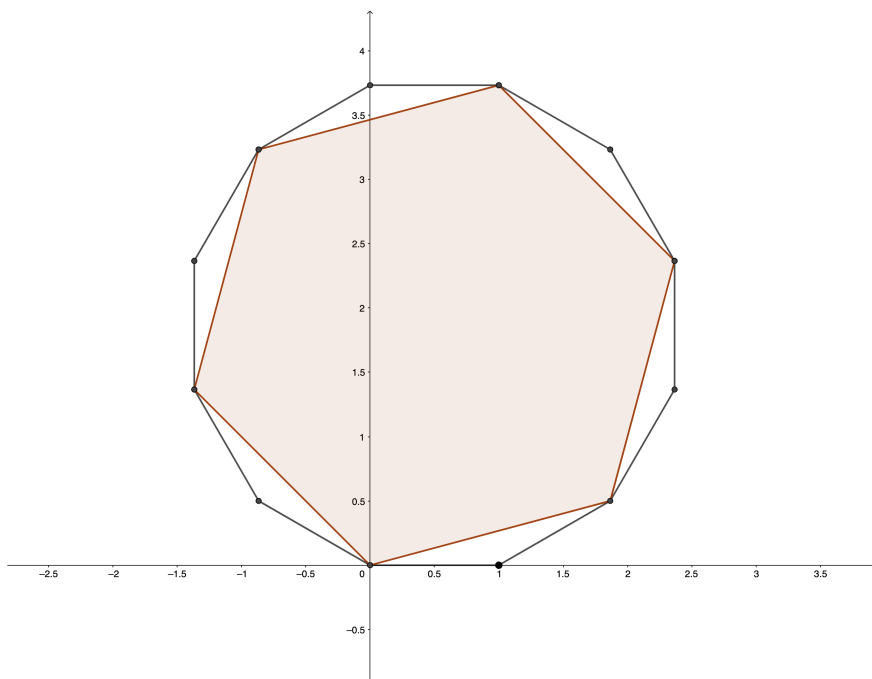


Figure 7: Hexagon inscribed in a dodecagon

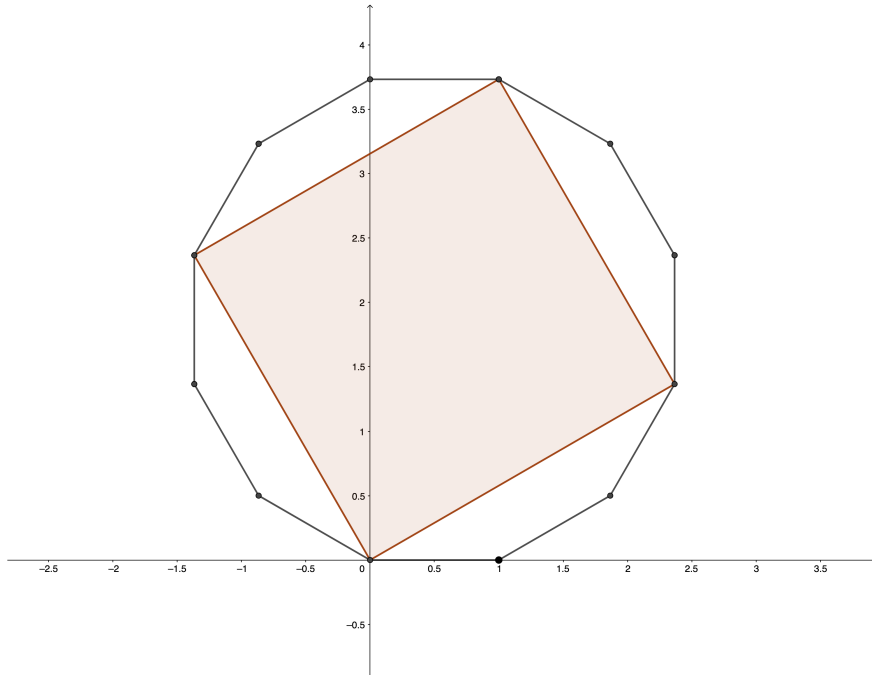


Figure 8: Square inscribed in a dodecagon

Factorising out for  $x - 1$ :

$$p_{12}(x) = \underbrace{(x - 1)}_{\text{number of vectors to sum}} \cdot \underbrace{(x^{10} + x^8 + x^6 + x^4 + x^2 + 1)}_{p'_6(x) \text{ with } k=2}$$

We obtain the polynomial that results to be  $p'_6(x)$  with  $k = 2$  Likewise we can obtain a square:

$$p_{12}(x) = \underbrace{(x^2 - x + 1)}_{\text{number of vectors to sum}} \cdot \underbrace{(x^9 - x^6 + x^3 - 1)}_{p'_4(x) \text{ with } k=3}$$

and a triangle:

$$p_{12}(x) = \underbrace{(x^3 - x^2 + x - 1)}_{\text{number of vectors to sum}} \cdot \underbrace{(x^8 + x^4 + 1)}_{p'_3(x) \text{ with } k=4}$$

## 4 A well known theorem

In this chapter, using Lill's method, we are going to give an interpretation of the role of the discriminant ( $\Delta = b^2 - 4ac$ ) of a quadratic equation.

First of all we state the well know theorem:

**Theorem 6.** Given a quadratic equation  $ax^2 + bx + c = 0$  whit  $a, b, c$  real numbers and  $a \neq 0$ :

- The equation has two coincident real solutions (double solutions) iff the discriminant is zero.
- The equation has two distinct real solutions iff the discriminant is positive.
- The equation has no real solutions iff the discriminant is negative.

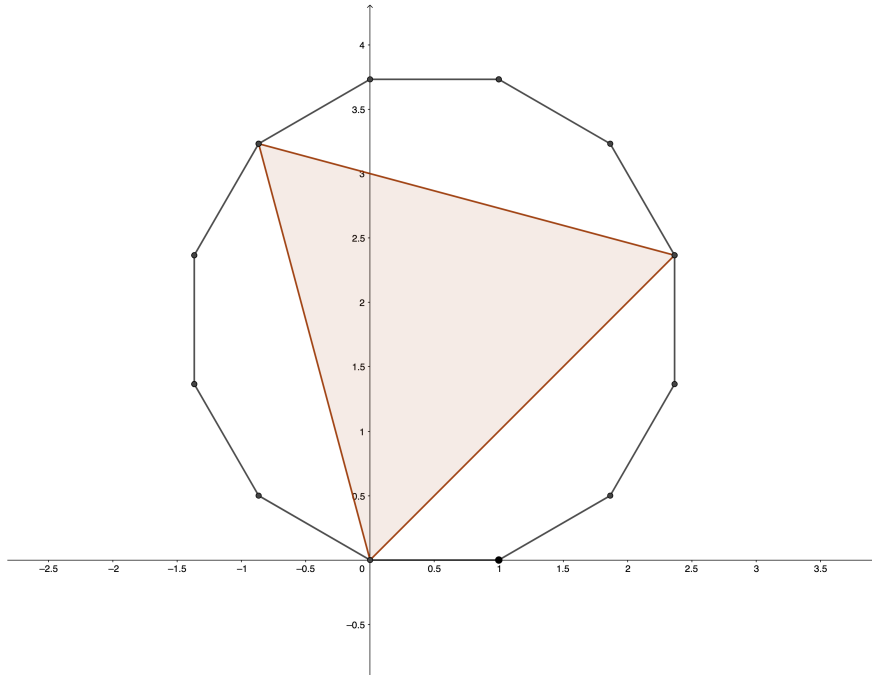


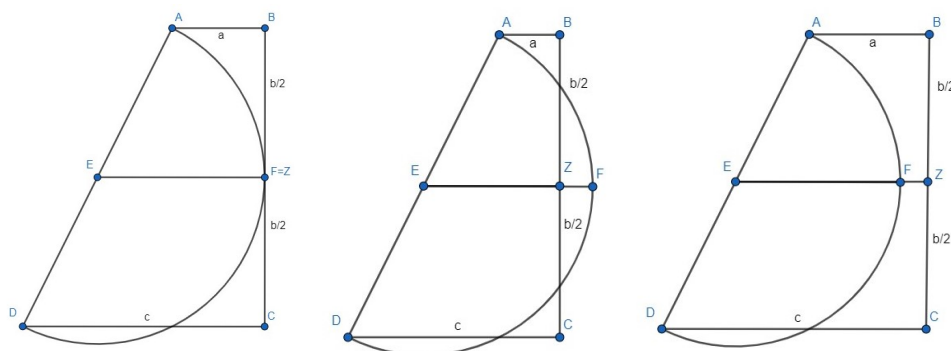
Figure 9: triangle inscribed in a dodecagon

[4]

*Proof.* We want to prove that if the equation has two coincident real solutions (or two or none) then the discriminant is zero (or positive or negative).

Let's consider, without loss of generality, a monic polynomial  $p(x) = x^2 + bx + c$  with  $b > 0$ ,  $c > 1$ .

Drawing the polynomial path  $ABCD$  of  $p(x)$ , the segment  $AD$  and the semicircle with diameter  $AD$ , we get the trapezoid  $ABCD$  and three different situations:



The geometrical construction in the first figure is related to the case in which the quadratic equation has one solution (double); in the central figure the quadratic equation has two different solutions; in the figure on the right the quadratic equation has zero solutions.

In all the three cases we can calculate the length of the segments  $EF$  and  $EZ$  parallel to the segments  $AB$  and  $CD$  joining the points  $E$  and  $Z$  respectively the middle point of  $AD$  and the middle point of  $BC$ .

The intersection between the segment  $BC$  and the semicircle represents the solution of the quadratic equation.

Starting from the analysis of the first figure, thanks to the Thales's theorem (also known as the intercept theorem) we can say that

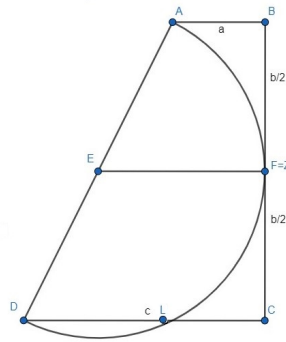
$$\overline{EZ} = \frac{\overline{AB} + \overline{DC}}{2} = \frac{a + c}{2}$$

We can see that the segment  $\overline{EF}$  is equal to the segment  $\frac{\overline{AD}}{2}$ .

If we consider  $Z$  and  $F$  coincident, as in the figure 1, we can say that  $\overline{EF} = \overline{EZ}$ , and therefore

$$\frac{\overline{AB} + \overline{CD}}{2} = \frac{\overline{AD}}{2} \Rightarrow \overline{AB} + \overline{CD} = \overline{AD}$$

Let's focus on the right triangle  $ADL$ .



According to Pythagorean theorem

$$\overline{DA}^2 = \overline{CB}^2 + (\overline{DC} - \overline{AB})^2$$

Since  $\overline{AB} + \overline{DC} = \overline{AD}$  we get

$$(\overline{AB} + \overline{DC})^2 = \overline{CB}^2 + (\overline{DC} - \overline{AB})^2$$

Solving the equation we get  $\overline{CB}^2 - 4(\overline{AB})(\overline{DC}) = 0$

We know that  $\overline{AB}$  corresponds to  $a$ ,  $\overline{BC}$  corresponds to  $b$  and  $\overline{CD}$  corresponds to  $c$  thus giving  $b^2 - 4ac = 0$  meaning that the discriminant is zero.

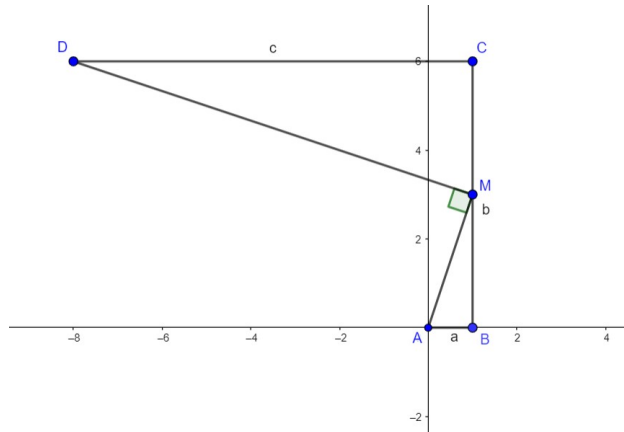
In the same way we can prove that if we consider  $F$  and  $Z$  like two non-correspondent points, if  $\overline{EZ}$  is less than  $\overline{EF}$ , as in the figure 2, then we have  $b^2 - 4ac > 0$  and if  $\overline{EZ}$  is greater than  $\overline{EF}$ , as in the figure 3, then we have  $b^2 - 4ac < 0$ .

□

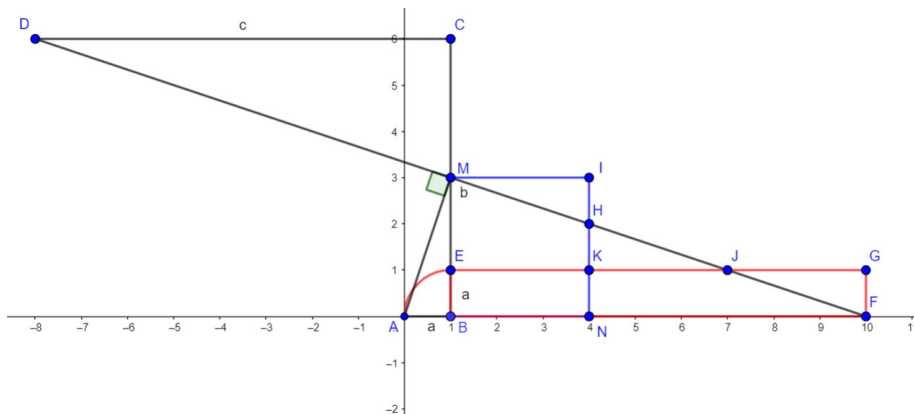
Now, we are going to propose a different proof of the same theorem, in the case of one solution, using the equidecomposability.

*Proof.* We consider an equation  $ax^2 + bx + c = 0$  with only one solution. We are going to prove that the discriminant is zero.

The polynomial path is



We build the square  $BNIM$  with side  $\frac{b}{2}$  and the rectangle  $BFGE$  with base  $c$  and height  $a$ :



To draw the rectangle we have to, for first, transfer  $\overline{AB}$  on  $\overline{BM}$  obtaining  $\overline{BE} = a$ . To transfer  $\overline{DC} = c$  on the  $x$  axis we have to extend  $\overline{DM}$  until  $x$  axis making the triangle  $BMF$ .

The triangle  $DMC$  is congruent to the triangle  $BMF$  because:

- $CM \cong BM$
- $\widehat{DMC} \cong \widehat{BMF}$  opposite at the vertex
- $\widehat{DCM} \cong \widehat{MBF}$  right angles

So  $\overline{DC} = \overline{BF} = c$

Now we are going to prove that the square  $BNIM$  is equivalent to the rectangle  $BFGE$  due to their equidecomposability.

First of all we say that  $ABM \cong MIH$  because:

- $BM \cong MI$
- $\widehat{ABM} \cong \widehat{MIH}$  (right angles)
- $\widehat{AMB} \cong \widehat{MHI}$  because they are both complementary to the angle  $\widehat{BMH}$ .



So  $\overline{AB} = \overline{IH} = a$ .

Also  $MIH \cong JGF$  because:

- $\overline{GF} = \overline{IH} = a$
- $H\widehat{MI} \cong F\widehat{JG}$  ( $MI$  and  $GE$  are parallel)
- $J\widehat{GF} \cong H\widehat{IM}$  (right angles)

Also  $HNF \cong MEJ$  because:

- $N\widehat{HF} \cong E\widehat{MJ}$  (because  $MB$  and  $NI$  are parallel)
- $\overline{ME} \cong \overline{NH}$  (differences of congruent sides)
- $E\widehat{JM} \cong N\widehat{FH}$  (because  $EG$  e  $BF$  are parallel)

So for the theorem of equidecomposability  $MEKH$  and  $NKJF$  are equivalent having  $HKJ$  in common.

Also  $MBNH$  and  $EBFJ$  are equivalent having  $BEKN$  in common.

At the end  $BMIN$  and  $BEFG$  are equivalent because  $MIH$  and  $FGL$  are congruent so equivalent.

The area of  $BMIN$  is  $\left(\frac{b}{2}\right)^2$  and the area of  $BEFG$  is  $ac$  then

$$\left(\frac{b}{2}\right)^2 = ac$$

that is the discriminant is zero. □

## 5 Geometrical constructions

In this section [\[5\]](#) we are going to analyse some geometrical applications of Lill's method.

### 5.1 Algebraic numbers

In this subsection we will deal with the algebraic numbers.

**Definition 4.** *An algebraic number is a real number for which exists a polynomial equation with integer coefficients such that the real number is a solution.*

Our goal consists to draw a segment whose measure is an algebraic number.

**Example 4.** *Let's analyse the algebraic number  $\sqrt[3]{2}$  from Delian problem.*

*First of all we find the polynomial equation  $x^3 - 2 = 0$  whose solution is the algebraic number, and, consequently, the polynomial  $p(x) = x^3 - 2$ .*

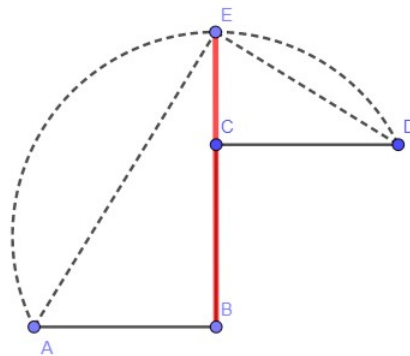
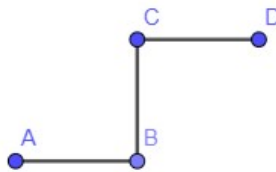
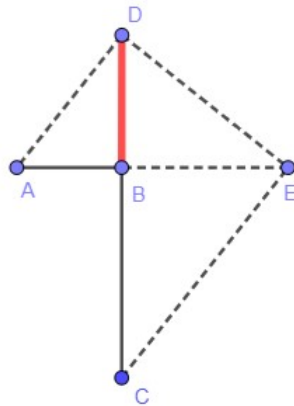
*Then we draw its polynomial path Extending the segment  $BC$  and drawing the solution path  $ADEC$  (dashed in the figure) we obtain the segment  $BD$  whose length is the algebraic number  $\sqrt[3]{2}$ .*

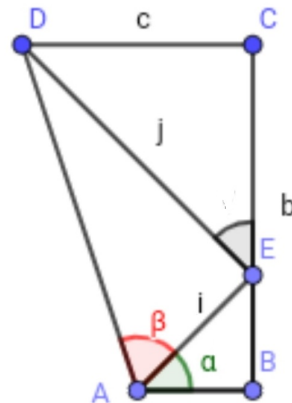
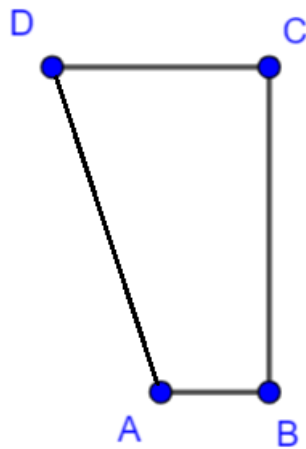
**Example 5.** *The same reasoning can be applied when the length of the segment that we have to find is the golden ratio  $\frac{1 + \sqrt{5}}{2}$ .*

*First of all we find the polynomial equation  $x^2 - x - 1 = 0$  whose solution is the algebraic number, and, consequently, the polynomial  $p(x) = x^2 - x - 1$ .*

*Then we draw its polynomial path*

*Extending the segment  $BC$  and drawing the solution path  $AED$  (dashed in the figure) we obtain the segment  $BE$  whose length is the algebraic number  $\frac{1 + \sqrt{5}}{2}$ .*





## 5.2 Angles

In this section we will deal with angles which are the sum of arctangents. Our goal consists to draw an angle whose measure is the sum of, in the first case, two arctangents and then, in the second one, three arctangents.

### First case.

Let's consider first the goal to draw an angle whose measure is  $\arctan(x_0) + \arctan(x_1)$  with  $x_0$  and  $x_1$  real positive numbers.

First we consider the polynomial  $p(x) = (x - x_0)(x - x_1) = x^2 - (x_0 + x_1)x + x_0x_1$ , whose solutions are  $x_0$  and  $x_1$ . Then we draw its polynomial path:

the angle  $D\hat{A}B$  measure  $\arctan(x_0) + \arctan(x_1)$ .

Let's prove this construction.

We suppose that  $x_0$  and  $x_1$  are positive so in the polynomial path  $a = 1$ ,  $b < 0$  and  $c > 0$ .

We observe that  $EB = x_0$  and  $CE = x_1$ .

Linking  $A$  and  $D$  we get the segment  $DA$  and the angle  $D\hat{A}B$ .

The measure of the angle corresponds to the sum of the angles  $B\hat{A}E$  plus  $E\hat{A}D$ . Considering the right triangle  $ABE$  we can say that  $\tan(\alpha) = EB = x_0$  so  $\alpha = \arctan(x_0)$ .

Now, looking at the right triangle  $AED$  we can say that  $\tan(\beta) = \frac{ED}{AE} = \frac{j}{i}$ .

But, since  $i = \frac{x_0}{\sin \alpha}$  and  $j = \frac{x_0 x_1}{\sin \alpha}$ , we find that  $\tan(\beta) = x_1$  therefore  $\beta = \arctan(x_1)$

Now we can conclude that

$$D\hat{A}B = \alpha + \beta = \arctan(x_0) + \arctan(x_1)$$

**Example 6.** We want to draw an angle whose measure is given by the sum of  $\arctan(1) + \arctan(2)$ .

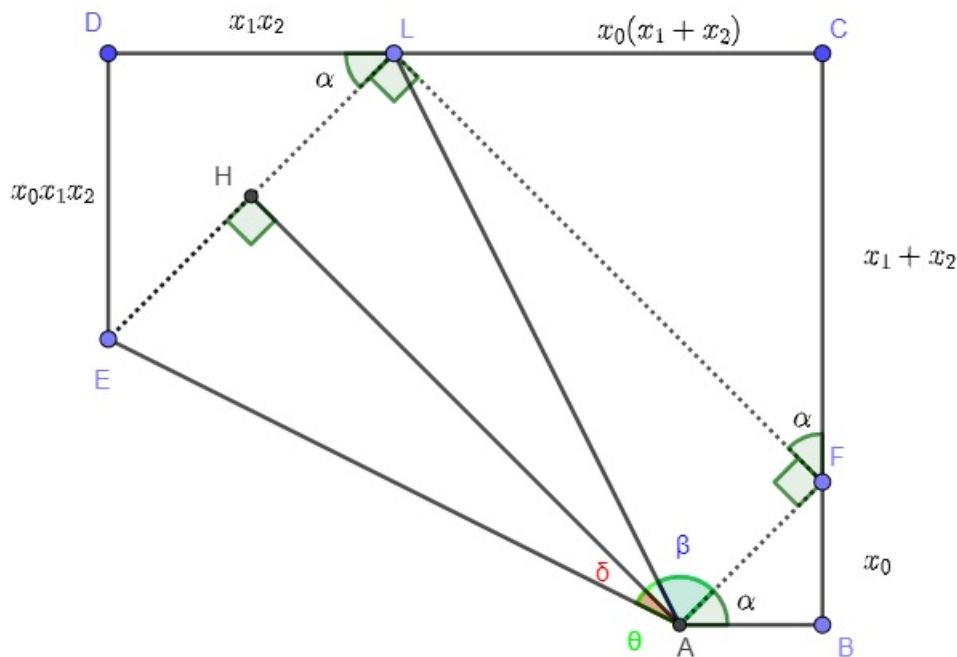
First we consider the polynomial  $p(x) = (x - 1)(x - 2) = x^2 - 3x + 2$ , then we draw its polynomial path  $ABCD$  with  $AB = 1$ ,  $BC = 3$  and  $CD = 2$ .

The angle  $D\hat{A}B$  measure  $\arctan(1) + \arctan(2)$ .

**Second case.**

Let's consider now the goal to draw an angle whose measure is  $\arctan(x_0) + \arctan(x_1) + \arctan(x_2)$  with  $x_0$ ,  $x_1$  and  $x_2$  real positive numbers.

Let's consider the polynomial  $p(x) = x^3 - (x_0 + x_1 + x_2)x^2 + (x_0x_1 + x_0x_2 + x_1x_2)x - x_0x_1x_2$  and its polynomial path, as in the image below:



We call  $\delta$  the angle  $E\hat{A}H$ ,  $\beta$  the angle  $L\hat{A}F$  and  $\theta$  the angle  $E\hat{A}F$ . Considering the right triangle  $LCF$  we note that  $LF = \frac{x_0(x_1 + x_2)}{\sin \alpha}$ . Considering the right triangle  $EDL$  we note that  $x_0x_1x_2 = EH \sin \alpha + HL \sin \alpha = EH \sin \alpha + x_0$  so  $EH = \frac{x_0(x_1x_2 - 1)}{\sin \alpha}$ . Considering the right triangle  $EHA$  we note that

$$\tan \delta = \frac{EH}{LF} = \frac{x_1x_2 - 1}{x_1 + x_2}$$

Now

$$\tan \theta = \tan \left( \delta + \frac{\pi}{2} \right) = -\cot \delta = \frac{x_1 + x_2}{1 - x_1 x_2}$$

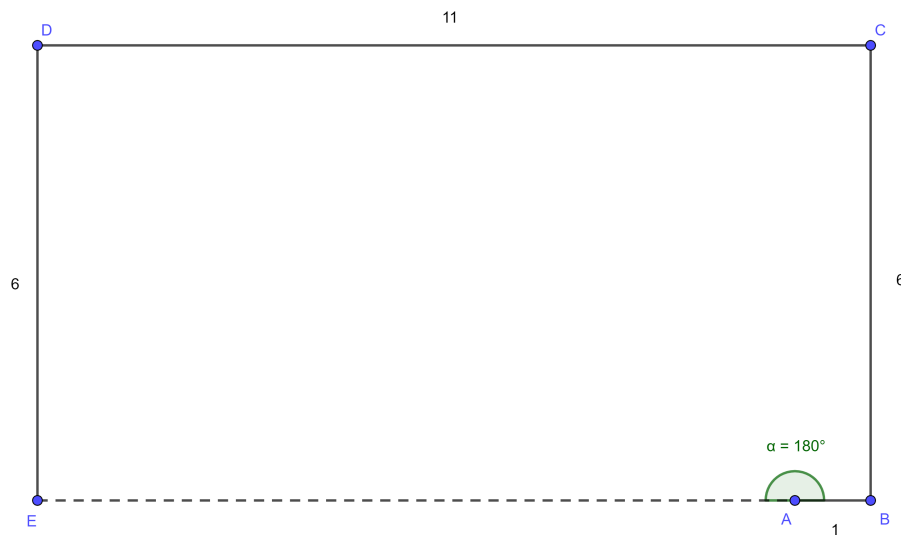
so

$$\theta = \arctan \frac{x_1 + x_2}{1 - x_1 x_2} = \arctan x_1 + \arctan x_2$$

We know that  $\alpha = \arctan x_0$  therefore

$$E\hat{A}B = \alpha + \theta = \arctan x_0 + \arctan x_1 + \arctan x_2$$

**Example 7.** An interesting example of this construction is when  $x_0 = 1$ ,  $x_1 = 2$  and  $x_2 = 3$ . In fact the polynomial becomes  $p(x) = x^3 - 6x^2 + 11x - 6$  and the polynomial path



and we can see that the measure of the angle  $E\hat{A}B$  is  $\arctan(1) + \arctan(2) + \arctan(3) = \pi$ .

[6]

## Edition notes

[1] This approximation of  $\pi$  is not done in this paper.

[2] When plotting associated polygons, the dominant coefficient is used as the unit. For unitary polynomials (which is the case of those treated in the article) this is not a problem. On the other hand, following the method of the article for  $4x^2 - 12x + 9$  it does not work. This poses a problem in particular in understanding the proof on page 24. It is mentioned that the proof will be given for all polynomials, this is only the case for those of degree 2 or 3.

[3] Theorem 3 (Lill's result) is stated in a very formal framework (any degree and polygon associated to a polynomial for any angle measure). In the proof there is only mention of degree 2 with any angle measure and degree 3 for an angle measure of  $\pi/2$ .

[4] The proof of Theorem 6 is done only for polynomials of the form  $x^2 + bx + c$  with  $b > 0$  and  $c > 1$  mentioning that it can be done without loss of generality. This deserves some explanation.

[5] In section 5 Lill's method is applied to construct segments of irrational length. The problem is that one has to succeed in tracing the path of the solutions to get there. This can be done approximately using geogebra software but absolutely not by hand.

[6] The article uses mathematics of a good terminal level following expert math. The results shown are interesting.