Tiling design

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1. Presentation of the research topic

A tiling design is a way of arranging various plane shapes (tiles) so that they completely cover a given area without overlapping and with no gaps. Despite their playful appearance, tiling problems have relevance to architecture and decorative arts, computer graphics, but also to Statistical Mechanics, where tilings can be used as models for molecule arrangements on a lattice. For example, if one considers a system in which a lattice is covered by monomers (basically a single molecule, modelled by 1×1 squares), dimers (a bond of two structurally similar monomers, modelled by dominos), trimers (a combination of three monomers, modelled by trominos) etc, then the thermodynamical properties of the system can be derived from the number of arrangements that can be found at its steady states (states with zero energy).

Our work focuses on counting the number of different tiling designs that can be obtained on a 2×10 or a 3×10 lattice, using three types of tiles: 1x1 squares, dominos and L-shaped trominos. To this end, we have derived appropriate recurrence relations, based on which we were able to find the number of all possible tiling designs. For all items in the research proposal, we have provided detailed combinatorial solutions, with suggestive graphical representations. Similar work can be done for more general lattices, such as chessboards or non-rectangular shaped lattices.

2. The topic

A 2×10 rectangular wall is going to be covered with tiles.
We have at our disposal three types of tiles, as shown in the figure below.

The tiles are unlimited and can be rotated in any way before being assembled.

**a)** In how many ways can we cover the wall only with domino tiles?

**b)** In how many ways can we cover the wall if the available tiles are of two types: type $1 \times 1$ and type $2 \times 1$?

**c)** What happens if we have all three types of tile available as drawn above?

**d)** In how many ways can we cover a $3 \times 10$ rectangular wall only with domino tiles?

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### 3. The Solution

**a)** Let us solve the problem from a) in the $2 \times n$ case (2 lines and $n$ columns). Let us consider $a_n$ as the number of ways in which we can completely cover, without overlapping, a $2 \times n$ wall with tiles $1 \times 2$.

We consider $a_0 = 1$. Because a $2 \times 1$ wall can be covered in only one way with a piece of tile $1 \times 2$ vertically placed as in the figure below, it follows that $a_1 = 1$.

For a $2 \times 2$ wall we have two possibilities, two $1 \times 2$ tiles, horizontally placed one above the other and two tiles $1 \times 2$, vertically placed:

So, $a_2 = 2$.

Let’s see now how can a $2 \times n$, $n \geq 3$, wall be covered.

– I. If we complete the first column with a $1 \times 2$ vertical tile, then the rest of the wall can be covered in $a_{n-1}$ ways.
If we cover the first two columns with two $1 \times 2$ tiles horizontally arranged, one under the other, then the rest of the wall will be covered in $a_{n-2}$ ways.

Because these are the only possibilities to begin the covering, it results that

$$a_n = a_{n-1} + a_{n-2}, \text{ for any natural number } n, \ n \geq 3.$$  

We observe that the formula is also correct for $n \geq 2$, with the convention that $a_0 = 1$.

It results that $a_2 = a_1 + a_0 = 2$, $a_3 = a_2 + a_1 = 3$, $a_4 = a_3 + a_2 = 5$, ..., $a_{10} = a_9 + a_8 = 89$.

In conclusion, the number of possibilities in which we can cover a $2 \times 10$ wall with $1 \times 2$ tiles is $a_{10} = 89$.

**Observation.** The formulas

$$\begin{cases} a_0 = a_1 = 1 \\ a_n = a_{n-1} + a_{n-2} \end{cases}$$

show that $(a_n)_{n \geq 0}$ is, in fact, the well-known Fibonacci sequence. It is known that (1)

$$a_{n-1} = F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right], \text{ for any natural number } n \geq 1.$$  

**b) Solution 1.**

Let us consider $b_n$ be the number of coverings of a $2 \times n$ wall with square $1 \times 1$ tiles or $1 \times 2$ dominoes.

We consider $b_0 = 1$. We observe that:

$b_1 = 2$

$$b_2 = 7$$

$$b_3 = 22$$
To find the values of $b_n$ for $n \geq 3$, we will use the next result.

**Theorem 1.** For every $n \geq 3$, we have

$$b_n = 3b_{n-1} + b_{n-2} - b_{n-3}.$$ 

**Proof.**

There are $b_{n-1}$ coverings of a $2 \times n$ wall which finish with two $1 \times 1$ square tiles in the $n^{th}$ column:

![Diagram of coverings ending with two 1x1 square tiles](diagram1)

There are $b_{n-1}$ coverings which finish with a $1 \times 2$ domino tile arranged vertically in the $n^{th}$ column:

![Diagram of coverings ending with a 1x2 domino tile](diagram2)

There are $b_{n-2}$ coverings which finish with two $1 \times 1$ square tiles arranged on the upper row and a horizontal $1 \times 2$ domino tile on the lower row, in the $(n-1)^{th}$ and $n^{th}$ column:

![Diagram of coverings ending with two 1x1 square tiles and a 1x2 domino tile](diagram3)
The remaining coverings of a $2 \times n$ wall can finish in the following ways:

Let us denote the above coverings with $(\ast)$. We will show that the number of $(\ast)$ coverings is $b_{n-1} - b_{n-3}$.

A covering of $2 \times (n - 1)$ can be finished in the following ways:

Each of the last three coverings of a $2 \times (n - 1)$ wall can be transformed, uniquely, in a covering of a $2 \times n$ wall of $(\ast)$ form [2]. So, the number of coverings of the $(\ast)$ form is $b_{n-1} - b_{n-3}$.
Using Theorem 1, we get \( b_0 = 1 \), \( b_1 = 2 \), \( b_2 = 7 \), \( b_3 = 3b_2 + b_1 - b_0 = 21 + 2 - 1 = 22 \) (we have also obtained this result by direct counting), \( b_4 = 3b_3 + b_2 - b_1 = 66 + 7 - 2 = 71 \), \( b_5 = 3b_4 + b_3 - b_2 = 213 + 22 - 7 = 228 \), ..., \( b_{10} = 3b_9 + b_8 - b_7 = 78243 \).

In conclusion, the number of possibilities in which we can cover a 2×10 wall with 1×2 domino tiles and 1×1 square tiles is \( b_{10} = 78243 \).

\textit{b) Solution 2.}

- Let \( x_n \) be the number of coverings of a 2×\( n \) wall ended in this way:

- Let \( y_n \) be the number of coverings of a 2×\( n \) wall ended in this way:

- Let \( z_n \) be the number of coverings of a 2×\( n \) wall ended in this way:
Let $c_n$ be the number of ways we can tile a $2 \times n$ wall with the following shapes:
Let $x_n$ be the number of coverings that can end the following ways:

\[ \begin{array}{c}
\text{or} \\
\end{array} \]

Let $y_n$ be the number of coverings that can end the following ways:

\[ \begin{array}{c}
\text{or} \\
\end{array} \]

Let $z_n$ be the number of coverings whose ends don’t fit anywhere above:

\[ \begin{array}{c}
\text{but aren’t counted in } y_n. \\
\end{array} \]
We denote with $c_n$ the number of coverings which finish like this:

\[
\begin{align*}
    c_n &= x_n + y_n + z_n, \text{ for any } n \geq 1 \\
    x_n &= 3c_{n-2} + z_{n-1}, \text{ for any } n \geq 3 \\
    y_n &= 2c_{n-1}, \text{ for any } n \geq 2 \\
    z_n &= 4c_{n-2} + z_{n-1} \text{ for any } n \geq 3
\end{align*}
\]

Using these formulas, we can find the value of $c_{10}$:

\[
\begin{align*}
    x_1 &= 0, \quad y_1 = 2, \quad z_1 = 0 \quad \Rightarrow \quad c_1 = 2 \\
    x_2 &= 3, \quad y_2 = 4, \quad z_2 = 4 \quad \Rightarrow \quad c_2 = 11 \\
    x_3 &= 10, \quad y_3 = 22, \quad z_3 = 12 \quad \Rightarrow \quad c_3 = 44 \\
    x_4 &= 45, \quad y_4 = 88, \quad z_4 = 56 \quad \Rightarrow \quad c_4 = 189 \\
    x_5 &= 188, \quad y_5 = 378, \quad z_5 = 232 \quad \Rightarrow \quad c_5 = 798 \\
    x_6 &= 799, \quad y_6 = 1596, \quad z_6 = 988 \quad \Rightarrow \quad c_6 = 3383 \\
    x_7 &= 3382, \quad y_7 = 6766, \quad z_7 = 4180 \quad \Rightarrow \quad c_7 = 14328 \\
    x_8 &= 14329, \quad y_8 = 28656, \quad z_8 = 17712 \quad \Rightarrow \quad c_8 = 60697 \\
    x_9 &= 60696, \quad y_9 = 121394, \quad z_9 = 75024 \quad \Rightarrow \quad c_9 = 257114 \\
    x_{10} &= 257115, \quad y_{10} = 514228, \quad z_{10} = 317812 \quad \Rightarrow \quad c_{10} = 1089155.
\end{align*}
\]

From (2) we can deduce a recurrence formula for $(c_n)_{n \geq 1}$. Indeed, we have:

\[
c_n = x_n + y_n + z_n = 3c_{n-2} + z_{n-1} + 2c_{n-1} + 4c_{n-2} + z_{n-1} = 2c_{n-1} + 7c_{n-2} + 2z_{n-1}, \text{ for any } n \geq 3,
\]

thus

\[
c_{n+1} - c_n = 2c_n + 5c_{n-1} - 7c_{n-2} + 2(z_n - z_{n-1}) = 2c_n + 5c_{n-1} - 7c_{n-2} + 8c_{n-2} = 2c_n + 5c_{n-1} + c_{n-2}, \text{ for any } n \geq 3,
\]

so

\[
c_{n+1} = 3c_n + 5c_{n-1} + c_{n-2}, \text{ for any } n \geq 3,
\]

or

\[
c_n = 3c_{n-1} + 5c_{n-2} + c_{n-3}, \text{ for any } n \geq 4,
\]

which, knowing that $c_1 = 2$, $c_2 = 11$ and $c_3 = 44$, gives us $c_{10} = 1089155$.

d) Solution 1

Let $d_n$ be the number of ways we can cover a $3 \times n$ rectangular wall with $2 \times 1$ domino tiles.

We denote with $x_n$ the number of coverings which finish like this:
We denote with $y_n$ the number of coverings which finish like this:

We have

\[
\begin{align*}
  d_n &= x_n + y_n, \text{ for any } n \geq 1 \\
  x_n &= 3 \cdot d_{n-2}, \text{ for any } n \geq 3 \\
  y_n &= 2d_{n-4} + y_{n-2}, \text{ for any } n \geq 5
\end{align*}
\]

Using (3), we find

\[
\begin{align*}
  x_1 &= 0, \ y_1 = 0 \Rightarrow d_1 = 0 \\
  x_2 &= 3, \ y_2 = 0 \Rightarrow d_2 = 3 \\
  x_3 &= 0, \ y_3 = 0 \Rightarrow d_3 = 0 \\
  x_4 &= 9, \ y_4 = 2 \Rightarrow d_4 = 11 \\
  x_5 &= 0, \ y_5 = 0 \Rightarrow d_5 = 0 \\
  x_6 &= 33, \ y_6 = 8 \Rightarrow d_6 = 41 \\
  x_7 &= 0, \ y_7 = 0 \Rightarrow d_7 = 0 \\
  x_8 &= 123, \ y_8 = 30 \Rightarrow d_8 = 153 \\
  x_9 &= 0, \ y_9 = 0 \Rightarrow d_9 = 0 \\
  x_{10} &= 459, \ y_{10} = 112 \Rightarrow d_{10} = 571.
\end{align*}
\]

Let us find now a recurrence formula for the sequence $\left( d_n \right)_{n \geq 1}$.

From (3), we get:

\[
d_n = x_n + y_n = 3d_{n-2} + 2d_{n-4} + y_{n-2}, \text{ for any } n \geq 5,
\]

thus

\[
d_{n+2} = 3d_n + 2d_{n-2} + y_n, \text{ for any } n \geq 3,
\]

therefore

\[
d_{n+2} - d_n = 3d_n + 2d_{n-2} + y_n - 3d_{n-2} - 2d_{n-4} - y_{n-2} = 3d_n - d_{n-2} - 2d_{n-4} + (y_n - y_{n-2}) = 3d_n - d_{n-2} - 2d_{n-4} + 2d_{n-4} = 3d_n - d_{n-2},
\]

so

\[
d_{n+2} = 4d_n - d_{n-2}, \text{ for any } n \geq 3.
\]

Finally, we have

\[
d_n = 4d_{n-2} - d_{n-4}, \text{ for any } n \geq 5.
\]

With this formula, and with $d_1 = d_3 = 0$, $d_2 = 3$, $d_4 = 11$, we can compute $d_n$, for any natural number $n$. This way, we get $d_{10} = 571$. 

\[
\]
**d) Solution 2.**

Let $d_n$ be the number of ways we can put the domino tiles, so we can have a complete $3 \times n$ wall.

![Diagram](image1)

Let $x_n$ be the number of ways we can arrange the domino tiles, so we have a $3 \times n$ wall, but with the top right corner missing.

![Diagram](image2)

Let $y_n$ be the number of ways we can arrange the domino tiles, so we have a $3 \times n$ wall, but with the bottom right corner missing.

![Diagram](image3)

These are all the possible ways the last column can end, any other possibility leading to a configuration with an infinite number of domino tiles.$^5$

![Diagram](image4)

All the possibilities from which we can obtain a $d_n$ type of tiling based on pre-calculations are:

![Diagram](image5)
So,
\[ d_n = d_{n-2} + x_{n-1} + y_{n-1}, \text{ for any even } n \geq 4. \]

**Observation.** We only have solutions for an even \( n \) in \( d_n \), respectively an odd \( n \) in \( x_n \) and \( y_n \).

All the possibilities in which we can obtain a \( x_n \) type of tiling based on pre-calculations (and not counted twice) are:

So,
\[ x_n = d_{n-1} + x_{n-2}, \text{ for any odd } n \geq 3. \]

**Observation:** \( x_n = y_n \), for any odd \( n \), because every arrangement counted in \( x_n \) can be associated with the same configuration from \( y_n \), but flipped horizontally.

We have the final formulas:
\[
\begin{cases}
    d_n = d_{n-2} + 2x_{n-1}, \text{ for any even } n \geq 4 \\
    x_n = d_{n-1} + x_{n-2}, \text{ for any odd } n \geq 3
\end{cases}
\]
with \( d_1 = 0, \; d_2 = 3, \; x_1 = 1 \).

From the first equation of (4), we get
\[
d_{n+1} - d_{n-1} = d_{n-1} + 2x_n - d_{n-3} = d_{n-1} - d_{n-3} + 2(x_n - x_{n-2}) = d_{n-1} - d_{n-3} + 2d_{n-1} = 3d_{n-1} - d_{n-3},
\]
thus
\[ d_{n+1} = 4d_{n-1} - d_{n-3}, \text{ for any } n \geq 3, \]
or
\[ d_n = 4d_{n-2} - d_{n-4}, \text{ for any } n \geq 5, \]
which is the same relation as in Solution 1.

The answer is \( d_{10} = 571 \).

4. Conclusion

We have answered all the questions in the research proposal, by providing detailed solutions and drawing suggestive diagrams. For some of the questions, we have given two different solutions. The solutions are based on the construction of relevant recurrence relations for the number of tiling designs. The recurrences we have derived in this paper are of orders two (for question (a)), three (for questions (b) and (c)) and four (for question (d)). We can see that, the more different types of tiles are used in the pattern design, the higher is the order of the recurrence relation. We expect that, for more complicated lattice shapes, the recurrence relation will have even higher orders, and their derivation will not be easy at all.
A proof of this result can be found for example in Wikipedia.

These transformations are shown in the figure above: one or two tiles are moved and a domino is added.

The formula for $z_n$ is justified as follows: we have two cases depending on whether the white square on line $n - 1$ is covered by a $1 \times 1$-tile or by a domino. In the first case, the $2 \times n$ covering is obtained from a $2 \times (n - 2)$ covering by adding two single squares on one row and one domino on the other (two possibilities); in the other case it is obtained from a covering of type $z_{n-1}$ by moving the isolated gray square of column $n - 1$ to column $n$ on the other row, and adding a domino (one possibility).

Note that for a covering of type $z_n$, the white square of column $n$ must be covered by either a domino or a tromino, since the case of two $1 \times 1$-tiles on the $n$-th column is counted in $y_n$.

Coverings of type $x_n$ ending with two dominos or a tromino and a $1 \times 1$-tile are obtained by completing a $2 \times (n - 2)$ covering (3 possibilities for each $2 \times (n - 2)$ covering); the others ones end with a tromino but the remaining square on column $n - 1$ is covered by a domino or a tromino: they correspond to a covering of type $z_{n-1}$ by replacing the last tromino by a $1 \times 1$-tile. The formula $x_n = 3c_{n-2} + z_{n-1}$ follows.

A similar argument shows that there are 4 ways to extend a $2 \times (n - 2)$ covering into a covering of type $z_n$; the coverings of type $z_n$ which cannot be obtained in this way (i.e., which cannot be cut after column $n - 2$) end with a domino on the row opposite the gray square, and the square next to the grey square on the same row is covered by a wider tile: they correspond to a covering of type $z_{n-1}$ by removing the domino and moving the gray $1 \times 1$-tile. So, we get $z_n = 4c_{n-2} + z_{n-1}$.

In fact, there is also the possibility of a single square at the top or bottom of the last column. But this has no consequence, since these configurations are not found in the proof below.