Equidecomposability of polygons

**Statement:** Two polygons A and B of equal area are given. Can A be cut into smaller polygons so that after rearrangement they form B? To make the redaction easier, we will use the following notations: A*B, which means that A can be cut into polygons which rearranged form B, and A!*B, which means that A cannot be cut into polygons which rearranged form B.

*Example 1:*

A= rectangle with sides 1 and 4

![Rectangle with sides 1 and 4]

A’= square with side 2

![Square with side 2]
Example 2:

A = square with side 2
B = square with side 3
A!*B

Remarks:

• If A*B, then B*A;
• If A*C and C*B, then A*B;

From these two remarks → if A*C and B*C, then A*B.

The actions of cutting and rearranging do not affect the area of the initial polygon. In order for A*B, it is absolutely necessary that the area of A is equal to the area of B.

We will prove that given two polygons A and B of equal areas, there is a way to cut A into smaller polygons that rearranged form B.

Steps:
I. A can be cut into triangles;
II. Any given triangle can be cut into smaller polygons that rearranged form a rectangle with equal area;
III. Any given rectangle can be cut into smaller polygons which rearranged form a rectangle with one side of length 1 and of equal area;
IV. All these rectangles with one side of length 1 can be put together to form a bigger rectangle with one side of length 1 and with the same area as A;

According to all these steps, we can say that $A \ast C$, for any polygon $A$. $B$ is also a polygon $\rightarrow B$ can also be cut into smaller polygons which rearranged form $C'$ (rectangle with one side of length 1 and the same area as $B$). Because $A$ and $B$ have the same area, $C$ and $C'$ are the same polygon. Then: $A \ast C$ and $B \ast C \rightarrow A \ast B$.

I. The triangulation of a polygon

For a convex polygon, it can be cut into triangles by choosing a vertex and tracing all of the diagonals which start from that vertex. For a concave polygon, it can be cut into convex polygons, and we repeat the previous steps.

II. Transforming any given triangle in a rectangle

a) We will transform the triangle $\Delta ABC$ in the rectangle $ZYCB$. We take $X$ and $Y$ as the midpoints of the line segments $[AB]$, respectively $[AC]$. We take $d$ as the parallel to $AC$, that goes through $B$, and $Z$ is the intersection of $d$ and $XY$ ($\{Z\} = d \cup XY$). $XY$ is the midsegment in $\Delta ABC$, $XY \parallel BC$. $XY \parallel BC$, $ZB \parallel YC \rightarrow ZYCB$ is a parallelogram. Because we know that $XYCB$ is $ZYCB$ intersected
with $\Delta ABC$ ($XYCB = ZYCB \cap \Delta ABC$), in order to show that $\Delta ABC \equiv ZYCB$, it suffices to prove that $\Delta ZX \equiv \Delta YX$.

$X$ is the midpoint of the line segment $[AB]$ ($X = \text{mid}[AB]$) $\rightarrow$ $AX \equiv BX$ (1), $\angle ZBX$ and $\angle YXA$ are opposite angles $\rightarrow$ $\angle ZBX \equiv \angle YXA$ (2); $ZB \parallel AY, AB = \text{transversal} \rightarrow \angle ZBX \equiv \angle YAX$ (alternate interior angles) (3). From (1), (2) and (3) $\Delta ZX \equiv \Delta YX$ (angle, side, angle - ASA)

b) We will transform the parallelogram $ZYCB$ into a rectangle $ZZ'YY'$. $Z'$ is the foot of the perpendicular from $Z$ to $BC$, and $Y'$ is the foot of the perpendicular from $Y$ to $BC$.

$ZZ' \perp BC; BC || ZY \rightarrow ZZ' \perp ZY \rightarrow ZZ' = \text{dist}(BC, ZY)$ - ($ZZ'$ is the distance from $BC$ to $ZY$)

In the same way we prove that $YY'$ is also the distance from $BC$ to $ZY$. ($YY' = \text{dist}(BC, ZY)$) $\rightarrow ZZ' = YY'$ (1)

$\alpha = m(\angle BZY)$
\[ \beta = m(\angle BZZ') \]
\[ \gamma = m(\angle CYY') \]
\[ \theta = m(\angle ZYC) \]

\[ BZ \parallel YC, ZY = \text{transversal} \Rightarrow \alpha + \theta = 180^\circ \, (i) \]
\[ \beta = \alpha - 90^\circ \, (ii) \]
\[ \theta + \gamma = 90^\circ \, (iii) \]

From (i), (ii) and (iii) \[ \alpha - \gamma = 90^\circ \Rightarrow \gamma = \alpha - 90^\circ \Rightarrow \gamma = \beta, \]
\[ \angle BZZ' = \angle CYY' \, (2) \]
\[ BZ = CY \, (3) \]

From (1), (2) and (3) \[ \Delta BZZ' \equiv \Delta CYY' \, (SAS - \text{side, angle, side}) \]

**III. The transformation of a rectangle into a rectangle with one side of length 1**

(1) We will prove that any rectangle \( X' \) can be transformed into a square with the same area \( \Rightarrow \) (2) any square can be transformed into any rectangle of the same area.

From (1) and (2) \( \Rightarrow \) any rectangle can be transformed into any other rectangle of the same area \( \Rightarrow \) any rectangle can be transformed into a rectangle of the same area and with one side of length 1.
We consider the rectangle $A'B'C'D'$, with sides of length $a'$ and $b'$ ($a'<b'$) and the square of equal area $XYZT$, with side of length $c=\sqrt{a' \cdot b'}$.

We will transform the rectangle $A'B'C'D'$ into the rectangle $ABCD$ with sides of lengths $a$ and $b$, and with the following properties:

\begin{align*}
a < b;\\
\frac{b}{c} \in (\sqrt{5}, \sqrt{10})
\end{align*}

(we will explain later on why this condition is necessary)

I. \[ \frac{b'}{c} \leq \sqrt{5} \]

We will do this transformation $\alpha$ times, with $\alpha \in \mathbb{N}$.

We cut the initial rectangle into another rectangle of sides $a' \cdot \frac{5}{6}$, respectively $b'$ and 5 rectangles of sides $a' \cdot \frac{1}{6}$ and $b' \cdot \frac{1}{5}$. These rectangles are then rearranged to form a rectangle with sides of length $a' \cdot \frac{5}{6}$ and $b' \cdot \frac{6}{5}$. 

\[ \frac{1}{5b'} \]

\[ 5/6a' \]

\[ b' \]

\[ 5/6a' \]

\[ b' \]

\[ 1/5b' \]

\[ 1/6a' \]
After the $\alpha$ transformations, $b$ will be equal to $b' \cdot \left(\frac{6}{5}\right)^\alpha$.

We will prove that $\exists \alpha \in \mathbb{N}$, so that $\frac{b'}{c} \cdot \left(\frac{5}{6}\right)^\alpha \in (\sqrt{5}, \sqrt{10})$.

Suppose that there isn’t any $\alpha \in \mathbb{N}$ that respects the property above $\to \exists \alpha' \in \mathbb{N}$ so that $\frac{b'}{c} \cdot \left(\frac{5}{6}\right)^{\alpha'} < \sqrt{10}$ and $\frac{b'}{c} \cdot \left(\frac{5}{6}\right)^{\alpha'} + 1 > \sqrt{5}$.

Let $x$ be $\frac{b'}{c} \cdot \left(\frac{5}{6}\right)^{\alpha'}$.

\begin{align*}
x \geq \sqrt{10} \Rightarrow x \cdot \frac{5}{6} \geq \sqrt{10} \cdot \left(\frac{5}{6}\right) & \quad (1) \\
x \cdot \frac{5}{6} \leq \sqrt{5} & \quad (2) \\
\sqrt{10} \cdot \left(\frac{5}{6}\right) \geq \sqrt{5} & \quad (3)
\end{align*}

From (1), (2) and (3) $\Rightarrow$ contradiction, so $\exists \alpha \in \mathbb{N}$ so that $\frac{b'}{c} \cdot \left(\frac{5}{6}\right)^\alpha \in (\sqrt{5}, \sqrt{10})$

II. $\frac{b'}{c} > \sqrt{10}$

We will do the following transformation $\alpha$ times, with $\alpha \in \mathbb{N}$.

We will cut the initial rectangle into another rectangle of sides $a'$, respectively $\left(\frac{1}{5}\right) \cdot b'$, and five rectangles of sides $\left(\frac{5}{12}\right) \cdot a'$, respectively $\left(\frac{5}{6}\right) \cdot b'$. These rectangles are rearranged in order to form a rectangle with sides of lengths $\left(\frac{6}{5}\right) \cdot a'$, respectively $\left(\frac{5}{6}\right) \cdot b'$. 
After the $\alpha$ transformations, $b$ will be equal to $b' \cdot \left(\frac{5}{6}\right)^{\alpha}$.

We will prove that $\exists \alpha \in \mathbb{N}$ so that $\left(\frac{b'}{c}\right) \cdot \left(\frac{5}{6}\right)^{\alpha} \in (\sqrt{5}, \sqrt{10})$.

Suppose that there is no $\alpha$ with the requested property

$\rightarrow \exists \alpha' \in \mathbb{N}$ so that $\left(\frac{b'}{c}\right) \cdot \left(\frac{5}{6}\right)^{\alpha'} + 1 \leq \sqrt{5}$, and $\left(\frac{b'}{c}\right) \cdot \left(\frac{5}{6}\right)^{\alpha'} \geq \sqrt{10}$.

Let $x$ be $\left(\frac{b'}{c}\right) \cdot \left(\frac{5}{6}\right)^{\alpha'}$.

$x \geq \sqrt{10} \rightarrow x \cdot \left(\frac{5}{6}\right) \geq \sqrt{10} \cdot \left(\frac{5}{6}\right)$ (1)

$x \cdot \left(\frac{5}{6}\right) \leq \sqrt{5}$ (2)

$\sqrt{10} \cdot \left(\frac{5}{6}\right) \geq \sqrt{5}$ (3)

From (1), (2) and (3) $\rightarrow$ contradiction. Therefore, $\exists \alpha \in \mathbb{N}$ so that $\left(\frac{b'}{c}\right) \cdot \left(\frac{5}{6}\right)^{\alpha} \in (\sqrt{5}, \sqrt{10})$. 
Let ABCD be a rectangle with sides of lengths a and b, respectively. XYZT a square with side of length \( c = \sqrt{a \cdot b} \) with the following properties:

\[
a < b;
\]

\[
\left( \frac{b}{c} \right) \in (\sqrt{5}, \sqrt{10})
\]

\( A_{ABCD} = A_{XYZT} \)

\( A'B'C'D' \) * ABCD

If we prove that \( ABCD * XYZT \rightarrow A'B'C'D' * XYZT \)

We overlap ABCD and XYZT so that \( D \in (YZ) \) and \( A \) is the same as \( X \).

In order to prove that \( ABCD * XYZT \), we will prove that there is a way to cut the rectangle ABCD so that the resulting polygons can form XYZT after rearrangement.

For this we will cut both XYZT and ABCD and we will prove that each polygon resulted from the dissection of the square is congruent with a polygon resulted from the dissection of the rectangle.

We will use the following notations:
\[ y = m(\angle YDA) \]
\[ F = BC \cap TZ \]
\[ H = \text{the foot of the perpendicular from } E \text{ to } BC \]
\[ N = \text{symmetric to } A \text{ with respect to } G \]
\[ K = \text{the foot of the perpendicular from } J \text{ to } YD \]
\[ L = \text{the foot of the perpendicular from } T \text{ to } AD \]
\[ l = \text{the length of the line segment } AE \]
\[ E = AD \cap TZ \]
\[ G = BC \cap AY \]
\[ I = BC \cap YD \]
\[ J = \text{symmetric to } A \text{ with respect to } E \]
\[ o = \text{the parallel to } BC \text{ through } N \]
\[ O = o \cap YZ \]
\[ d = \text{the length of the line segment } AG \]

XYZT is cut into \{AEFG; TLE; GFZON; NOY; TLA\}

ABCD is cut into \{AEFG; ABG; EJKIF; JDK; DCI\}

We will prove that:

(1) \[ TLE \equiv ABG \]
(2) \[ GFZON \equiv EJKIF \]
(3) \[ NOY \equiv IDK \]
(4) \[ TLA \equiv DCI \]
(5) \[ AEFG \equiv AEFG \]
From (1), (2), (3), (4) and (5) results that there is a way to cut ABCD into polygons that rearranged form XYZT

\[ AY \perp YD \rightarrow \Delta AYD \text{ is a right triangle } \rightarrow \sin(y) = \frac{AY}{AD} = \frac{c}{b} \]

\[ EZ \perp ZD \rightarrow \Delta EZD \text{ right triangle } \rightarrow m(\angle ZED) = 90^\circ - y \]
\[ EH \perp BC, AD \parallel BC \rightarrow EH \perp AD \rightarrow m(\angle HED) = 90^\circ \ (1) \]
\[ m(\angle HED) = m(\angle HEF) + m(\angle ZED) \ (2) \]
\[ m(\angle HEF) = y \ (3) \]
\[ EH \perp HF \rightarrow \Delta EHF \text{ is a right triangle } (4) \]

From (1), (2), (3) and (4) → \( \cos(y) = \frac{EH}{EF} \) (i)

\[ EH \perp BC, BC \parallel AD \rightarrow EH = \text{distance from BC to AD} \ (1) \]
\[ AB \perp BC, BC \parallel AD \rightarrow AB = \text{distance from BC to AD} \ (2) \]
From (1) and (2) → AB = EH = a (ii)

\[ EF \parallel AG, AE \parallel GF \rightarrow AEFG \text{ is a parallelogram } \rightarrow EF = AG = d \ (iii) \]

From (i), (ii) and (iii) → \( \cos(y) = \frac{a}{d} \)

\[ E' = \text{the foot of the perpendicular from E to AY} \]
\[ EE' \perp AY, ZT \parallel AY \rightarrow EE' = \text{the distance from AY to ZT} \rightarrow EE' = c \]
\[ ZT \parallel AY, YZ \perp AY \rightarrow YZ = \text{the distance from AY to ZT} \rightarrow YZ = c \]
\[ YZ = EE' = c \ (1) \]
\[ \Delta AE'E \text{ = right triangle } (2) \]

In \( \Delta AYD \):
\[ m(\angle AYD) = 90^\circ, m(\angle ADY) = y \rightarrow m(\angle YAD) = 90^\circ - y = m(\angle E'AE) \ (3) \]
From (2) and (3) → \( m(\angle AEE') = y \) (4)

From (1), (2) and (4) → \( \cos(y) = \frac{GE}{AE} = \frac{c}{l'}, \sin(y) = \frac{AE'}{l} \)

\[ A_{ABCD} = a \cdot b, \ A_{XYZT} = c^2 \rightarrow ab = \frac{c^2}{b \cdot c} \rightarrow \frac{a}{c} = \frac{c}{b} = \sin(y) \]
\[
\sin(y) \cdot \frac{\cos(y)}{\cos(y)} = \frac{a}{c} \cdot \frac{c}{l} \cdot \frac{d}{a} = \frac{d}{l} \rightarrow \sin(y) = \frac{AE'}{l}, \sin(y) = \frac{AE'}{l}
\]

\( \rightarrow AE' = d, \text{ and } G, E' \in [AY] \rightarrow E' = G \)

\( EG \perp AY \rightarrow EG \parallel YD \)

AEFG is a parallelogram \( \rightarrow EF = AG, AG = GN \rightarrow EF = GN \)

AE \parallel GF, AG transversal \( \rightarrow m(\angle GAE) = m(\angle NGF) = 90^\circ - y \), but

\( m(\angle FEJ) = 90^\circ - y \rightarrow \angle NGF \equiv \angle FEJ \)

EJ \parallel FI, EF transversal \( \rightarrow m(\angle FEJ) + m(\angle EFI) = 180^\circ \rightarrow m(\angle EFI) = 90^\circ + y \)

GF \parallel NO, GN transversal \( \rightarrow m(\angle NGF) + m(\angle GNO) = 180^\circ \)

\( \rightarrow m(\angle GNO) = 90^\circ + y \)

\( \angle EFI \equiv \angle GNO \)

\( \angle EFI \equiv \angle GFZ \) (opposite angles)

\( \Delta JKD \) is a right triangle, \( m(\angle KDJ) = y \rightarrow m(\angle KJD) = 90^\circ - y \)

\( \angle KJE \text{ and } \angle KJD \) are supplementary angles

\( \rightarrow m(\angle KJE) = 90^\circ + y = m(\angle EFI) \)

\( \angle KJE \equiv \angle EFI, \angle EFI \equiv \angle GFZ \rightarrow \angle KJE \equiv \angle GFZ \)

\( JK \perp YD, I \in YD \rightarrow m(\angle JKI) = 90^\circ \)

FZ \perp YD, O \in YD \rightarrow m(\angle FZO) = 90^\circ \)

NY \perp YZ, FZ \perp YZ \rightarrow NY \parallel FZ

YZ \equiv ZI (they’re actually the same line)

NY \parallel FZ, NO \parallel FI, YO \equiv ZI \rightarrow \triangle NYO \sim \triangle FZI \rightarrow \angle NOY \equiv \angle FIZ

\( \angle NOZ \text{ and } \angle NOY \) supplementary, \( \angle FIK \text{ and } \angle FIZ \) supplementary,

\( \angle NOY \equiv \angle FIZ \rightarrow FIK \equiv NOZ \)

\( \angle NGF \equiv \angle FEJ, \angle GNO \equiv \angle EFI, \angle GFZ \equiv \angle EJK, \angle FZO \equiv \angle JKI, \angle ZON \equiv \angle KIF \)

\( \rightarrow \triangle EJKIF \sim \triangle GFZON, EF = GN \rightarrow \triangle EJKIF \equiv \triangle GFZON \rightarrow JK = FZ, FI = NO, IK = OZ \)

NO \parallel BC, BC \parallel AD \rightarrow NO \parallel AD

NY \perp YD, JK \perp YD \rightarrow NY \parallel JK

NO \parallel AD, NY \parallel JK, YO \equiv KD \rightarrow \triangle NYO \sim \triangle JKD, \triangle NYO \sim \triangle FZI
\[
\Delta JKD \sim \Delta FZI, \ JK = FZ \Rightarrow \Delta JKD \equiv \Delta FZI \\
\Delta NYO \sim \Delta FZI, \ FI = NO \Rightarrow \Delta NYO \equiv \Delta FZI, \ \Delta JKD \equiv \Delta FZI \Rightarrow \Delta NYO \equiv \Delta JKD \\
\Rightarrow \ YO = KD \\
C = \text{YO} + \text{OZ} = KD + IK = ID \\
m(\angle GAD) = 90^\circ - y, \ \angle GAD \text{ and } \angle DAT \text{ complementary angles} \\
\Rightarrow m(\angle DAT) = y \\
\Delta FZI \equiv \Delta JKD \Rightarrow m(\angle FIZ) = m(\angle JDK) = y, \ m(\angle CID) = m(\angle FIZ) \text{ (opposite angles)} \Rightarrow m(\angle CID) = y = m(\angle LAT) \\
\Delta LAT \text{ and } \Delta CID \text{ are right triangles } \ AT = ID, \ \angle LAT \equiv \angle CID \\
\Rightarrow \Delta LAT \equiv \Delta CID \Rightarrow \ LT = CD \\
\text{ABCD = rectangle} \Rightarrow \ AB = CD = LT \\
\angle BGA \text{ and } \angle AGF \text{ are supplementary angles, } \angle LET \text{ and } \angle LEF \text{ are supplementary angles, } m(\angle AGF) = m(\angle LEF) = 90^\circ + y \Rightarrow \angle BGA \equiv \angle LET \\
\Delta ABG \text{ and } \Delta TLE = \text{ right triangles, } \ AB = LT \Rightarrow \Delta ABG \equiv \Delta TLE \\
\text{AEFG} \equiv \text{AEFG}, \ \text{ABG} \equiv \text{TLE}, \ \text{EJKIF} \equiv \text{GFZON}, \ \text{IDK} \equiv \text{NOY}, \ \text{DCI} \equiv \text{TLA} \\
\Rightarrow \text{ABCD} \ast \text{XYZT} \Rightarrow \text{A’B’C’D’} \ast \text{XYZT} \text{ and } \text{XYZT} \ast \text{A’B’C’D’}. \\
\]

Therefore, any rectangle can be transformed into a square of equal area, and a square can be transformed into any rectangle of equal area. In other words, any rectangle can be transformed into another rectangle of equal area.

In conclusion, this is the procedure (not necessarily the best one) to dissect a polygon A into smaller polygons which rearranged form B.

Recap:
1) We start by cutting A in triangles (we proved at I that it’s possible for any given polygon)
2) We transform each triangle firstly in a parallelogram of equal area, and then in a rectangle of the same area. (we proved at II that it’s possible for any triangle)
3) We transform each rectangle in a square of equal area, which will be then transformed into a rectangle with one side of length 1 (or any other length as long as it’s the same throughout the whole process) (we proved at III that it’s possible for any square). We put together all the rectangles with one side of length 1 to form a bigger rectangle (a) with one side of length 1 and the same area as A.

4) We repeat steps 1), 2) and 3) for polygon B, to obtain a rectangle with one side of length 1 and the same area as B (b). Because B and A have the same area, rectangle a and rectangle b will be the same rectangle. In order to obtain B from A, we repeat the steps in reverse order.

Now, here’s why we need $\frac{b}{c} \in (\sqrt{5}, \sqrt{10})$ in order to use this demonstration:

In order to consider the pentagon EJKIF, I needs to be more to the left of K (YI < YK), thus the line segments [JK], respectively [IF] intersect.

In ΔGIY we observe that YI = \text{ctg}(y) \cdot GY = \text{ctg}(y) \cdot (c-d), NY = JK, NY \parallel JK → NYKJ is a parallelogram → YK = NJ, YK \parallel NJ, YK \perp AY → NJ \perp AY → ΔANJ is a right triangle

\[ m(NAJ) = 90^\circ - y, \, m(JNA) = 90^\circ \rightarrow m(NJA) = y \]

→ NJ = \cos(y) \cdot AJ = \cos(y) \cdot 2l, YK = \cos(y) \cdot 2l

\[ \cos(y) \cdot 2l > \frac{\cos(y)}{\sin(y)} \frac{c-d}{\sin(y)} \]
\[
\cos(y) \cdot 2l \cdot \left(\frac{d}{l}\right) > \cos(y) \cdot (c-d)
\]

\[
2d > c-d \rightarrow 3d > c \rightarrow 3 > \frac{c}{d}, \quad d = \frac{a}{\cos(y)} \rightarrow 3 > c \cdot \frac{\cos(y)}{a}
\]

\[
\frac{c}{a} = \frac{b}{c} = \frac{1}{\sin(y)} \rightarrow 3 > \frac{\cos(y)}{\sin(y)} \rightarrow 3 > \cot(y)
\]

In order to consider the pentagon GNOZF, F needs to be higher than Z (F \in [TZ]). We will consider x to be the distance (with sign) from F to Z (positive if F is higher than Z, and negative if F is lower than Z).

\[
TE + EF + FZ = TZ
\]

\[
TE = EF = d, \quad TZ = c
\]

\[
2d + FZ = c, \quad \text{and we want } FZ > 0 \rightarrow 2d < c \rightarrow 2 < \frac{c}{d} \quad \text{and through the same steps} \quad 2 < \cot(y)
\]

\[
\cot(y) \in (2, 3), \quad y \in (0^\circ, 90^\circ), \quad \cot \text{ is a monotonically decreasing function on } (0^\circ, 90^\circ) \rightarrow y \in (\arctg(3), \arctg(2))
\]

\[
sin \text{ is a monotonically increasing function on } (0^\circ, 90^\circ)
\]

\[
\rightarrow \sin(y) \in (\sin(\arctg(3)), \sin(\arctg(2))), \quad \text{therefore } \sin(y) \in \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{5}}\right)
\]

\[
\sin(y) = \frac{c}{b} \rightarrow \frac{b}{c} \in (\sqrt{5}, \sqrt{10})
\]